Introduction to the theory of voting II

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February 12, 2020

Much more to talk about...

- Impossibility theorems
- Probabilistic social choice
- Voting by grading (Approval Voting, Majority Judgement, Score Voting)
- Alternative voting methods (quadratic voting, liquid democracy)
- Characterization results/Voting methods as statistical estimators
- Strategic voting
- Behavioral social choice

Suppose that X is a set of candidates and V a set of voters.

- Voting rule: $f : \mathcal{B}^V \to \wp(X) \setminus \emptyset$
  - $\mathcal{B}$ is the set of **ballots** (strict linear orders over X, strict weak orders over C, an assignment of *grades* to X)
  - A voting rule is **resolute** when all $\mathbf{B} \in \mathcal{B}^V$, $|f(\mathbf{B})| = 1$
  - Also called a **social choice function** or **social choice correspondence**

- Collective choice rule: $f : \mathcal{D} \to P(X)$
  - $\mathcal{D}$ is the domain of f (typically a subset of $O(X)^V$ where $O(X)$ is the set of strict weak orders over X)
  - The co-domain $P(X)$ is the set of binary relations on X
  - Also called a **social welfare function** (typically assume the range is complete, reflexive and transitive).

Define f_{maj} as follows: for all $\mathbf{P}$,

$$f_{maj}(\mathbf{P}) = \{(x, y) \mid |\mathbf{P}(x, y)| > |\mathbf{P}(y, x)|\}$$

(so, $f_{maj}(\mathbf{P}) = >_{\mathbf{P}}$).

Let $\mathbf{P}$ be the following profile (a Condorcet cycle):

1	1	1
a	b	c
b	c	a
c	a	b

$f_{maj}(\mathbf{P})$ is not transitive.

Plurality order

Define f_{pl} as follows: for all $\mathbf{P} \in L(X)^V$,

$$f_{pl}(\mathbf{P}) = \{(x, y) \mid PL_{\mathbf{P}}(x) \geq PL_{\mathbf{P}}(y)\}$$

Plurality order

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$$f_{pl}(\mathbf{P}) = \{(x, y) \mid PL_{\mathbf{P}}(x) \geq PL_{\mathbf{P}}(y)\}$$

$$\mathbf{P}: \quad \begin{array}{cc} 2 & 1 \\ \hline a & b \\ c & c \\ d & d \\ b & a \end{array}$$

not $c \; f_{pl}(\mathbf{P}) \; d$ even though $\mathbf{P}(c, d) = V$.

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$$f_{pl}(\mathbf{P}) = \{(x, y) \mid PL_{\mathbf{P}}(x) \geq PL_{\mathbf{P}}(y)\}$$

$\mathbf{P}$:

40	35	25
t	c	k
k	k	t
c	t	c

$\mathbf{P}'$:

40	35	25
t	k	k
k	t	t
c	c	c

$t\ f_{pl}(\mathbf{P})\ c\ f_{pl}(\mathbf{P})\ k$

$k\ f_{pl}(\mathbf{P}')\ t\ f_{pl}(\mathbf{P}')\ c$

$\mathbf{P}_{|\{k,t\}} = \mathbf{P}'_{|\{k,t\}}$

Borda order

Define f_{borda} as follows: for all $\mathbf{P} \in L(X)^V$,

$$f_{borda}(\mathbf{P}) = \{(x, y) \mid BS_{\mathbf{P}}(x) \geq BS_{\mathbf{P}}(y)\}$$

$\mathbf{P}$:

45	55
a	b
c	a
b	c

$\mathbf{P}'$:

45	55
a	b
b	a
c	c

$a\ f_{borda}(\mathbf{P})\ b\ f_{borda}(\mathbf{P})\ c$
$b\ f_{borda}(\mathbf{P}')\ a\ f_{borda}(\mathbf{P}')\ c$
$\mathbf{P}_{|\{a,b\}} = \mathbf{P}'_{|\{a,b\}}$

Preferences (Rankings)

asymmetry: if xPy, then *not* yPx;

negative transitivity: if xPy, then xPz or zPy.

Negative Transitivity

$$\text{if } xPy \text{, then } xPz \text{ or } zPy$$

Negative transitivity is equivalent to the condition that:

$$\text{if } \textit{not } xPz \text{ and } \textit{not } zPy \text{, then } \textit{not } xPy.$$

Negative Transitivity

$$\text{if } xPy, \text{ then } xPz \text{ or } zPy$$

Negative transitivity is equivalent to the condition that:

$$\text{if } \textit{not } xPz \text{ and } \textit{not } zPy, \text{ then } \textit{not } xPy.$$

Together negative transitivity and asymmetry imply that P is transitive:

transitivity: if xPy and yPz, then xPz.

Non-compariability

Let xNy if and only if neither xPy nor yPx. We call N the relation of *non-comparability*.

If P is a strict weak order, then N satisfies the following for all $x, y, z \in X$:

transitivity of non-comparability: if xNy and yNz, then xNz.

P is a *strict weak order* if and only if P satisfies asymmetry and negative transitivity

P is a *strict linear order* if and only if it satisfies asymmetry, transitivity, and weak completeness: for all $x, y \in X$, if $x \neq y$, then xPy or yPx.

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P is a *strict linear order* if and only if it satisfies asymmetry, transitivity, and weak completeness: for all $x, y \in X$, if $x \neq y$, then xPy or yPx.

$P(X)$ is the set of all asymmetric binary relations on X;
$O(X)$ is the set of all strict weak orders on X;
$L(X)$ is the set of all strict linear orders on X.

Profiles

A **profile P** is an element of $O(X)^V$, i.e., a function assigning to each $i \in V$ a relation $P_i \in O(X)$. For $x, y \in X$, let:

$$\begin{aligned} \mathbf{P}(x,y) &= \{i \in V \mid xP_iy\}; \\ \mathbf{P}_{|\{x,y\}} &= \text{the function assigning to each } i \in V \\ & \text{ the relation } P_i \cap \{x,y\}^2. \end{aligned}$$

A *collective choice rule* (CCR) for $\langle X, V \rangle$ is a function f from a subset of $O(X)^V$ to $P(X)$. By $xf(\mathbf{P})y$, we mean $\langle x, y \rangle \in f(\mathbf{P})$.

Domain Conditions

universal domain (UD): $\mathrm{dom}(f) = O(X)^V$.

linear domain (LD): $\mathrm{dom}(f) = L(X)^V$.

Codomain Conditions ("Rationality Postulates")

transitive rationality (TR): for all $\mathbf{P} \in \mathrm{dom}(f)$, $f(\mathbf{P})$ is transitive.

full rationality (FR): for all $\mathbf{P} \in \mathrm{dom}(f)$, $f(\mathbf{P})$ is a strict weak order.

Interprofile Conditions

independence of irrelevant alternatives (IIA): for all $\mathbf{P}, \mathbf{P}' \in \mathrm{dom}(f)$ and $x, y \in X$,

$$\text{if } \mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}, \text{ then } f(\mathbf{P})_{|\{x,y\}} = f(\mathbf{P}')_{|\{x,y\}}.$$

Decisiveness Conditions

Pareto (P): for all $\mathbf{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if $\mathbf{P}(x, y) = V$, then $xf(\mathbf{P})y$.

dictatorship: there is an $i \in V$ such that for all $\mathbf{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if xP_iy, then $xf(\mathbf{P})y$.

Theorem (Arrow 1952). Assume that $|X| \geq 3$ and V is finite. Then any CCR for $\langle X, V \rangle$ satisfying UD, IIA, FR, and P is a dictatorship.

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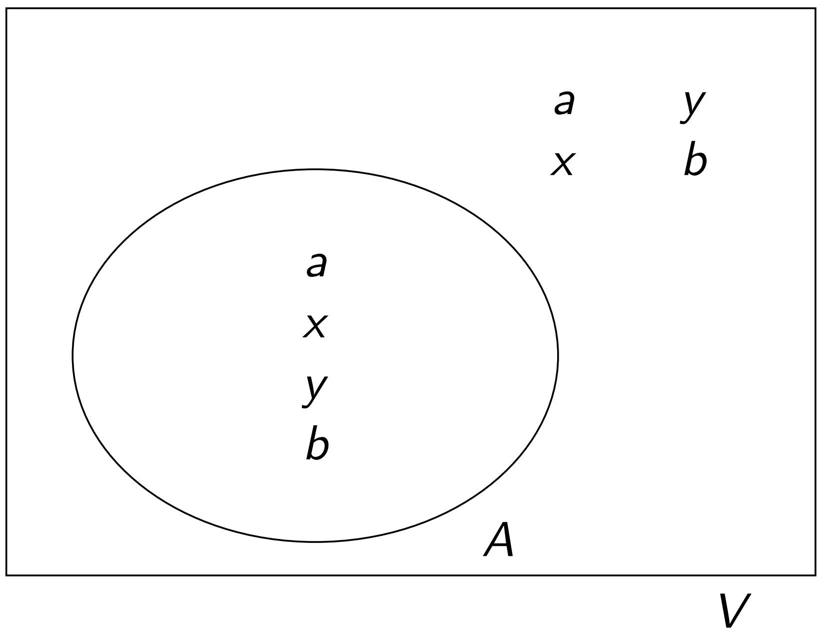
- Proof strategies: Pivotal voter proofs; The structure of decisive coalitions; Many generalizations
- What are the CCRs that Arrow's axioms *characterize*? (What exactly is a CCR that is dictatorial?)
- Cf., Mossel, E. (2012). A Quantitative Arrow Theorem. Probability Theory and Related Fields, 154 (1), 49-88.

Decisive Coalition

$A \subseteq V$ is **decisive** for x over y, if for all $\mathbf{P} \in dom(f)$, if $A \subseteq \mathbf{P}(x, y)$, then $x\ f(\mathbf{P})\ y$.

Decisiveness Spread Lemma. For any $A \subseteq V$ and candidates $x, y \in X$, if A is decisive for x over y, then for any $z, w \in X$, A is decisive for z over w.

Decisiveness Spread Lemma



1. Pareto implies that $a\ f(\mathbf{P}')\ x$
2. Pareto implies that $y\ f(\mathbf{P}')\ b$
3. A is decisive for x over y implies that $x\ f(\mathbf{P}')\ y$
4. $f(\mathbf{P}')$ is (quasi-)transitive, so:
    4.1 1. and 3. implies that $a\ f(\mathbf{P}')\ y$
    4.2 4(a). and 2. implies that $a\ f(\mathbf{P}')\ b$

Suppose that f is a CCR for (X, V). A set $A \subseteq V$ is an **oligarchy** for f if A is decisive for f and for $\mathbf{P} \in dom(f)$, if $x\mathbf{P}_i y$ for some $i \in A$, then not $y\ f(\mathbf{P})\ x$.

Gibbard's Oligarchy Theorem. Assume that $|X| \geq 3$ and V is finite. Then any CCR for $\langle X, V \rangle$ satisfying UD, IIA, TR, and P has an oligarchy.

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strict non-imposition (SNI): for all $x, y \in X$ with $x \neq y$, there is an $\mathbf{P} \in dom(f)$ such that $x\ f(\mathbf{R})\ y$.

inverse-dictator: d is an *inverse dictator* if for all $\mathbf{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if $x\mathbf{P}_i y$, then $y\ f(\mathbf{P})\ x$.

Murakami's Theorem (1968). Any CCR satisfying UD, FR, IIA and SNI is either dictatorial or inversely dictatorial.

An Example of Strategic Voting

1	2	3	4	5
c	a	b	c	a
b	c	a	b	c
a	b	c	a	b

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a wins

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1	2	3	4	5
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a	b	a	a	c

b wins

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b	a	b	b	a
a	b	a	a	c

b wins

Since **voter 1 prefers** b **to** a (top left), she has an incentive to report the insincere preference in blue to get a better outcome.

Manipulation of Resolute Rules

A **pointed profile** is a pair $(\mathbf{P}, i)$ where $\mathbf{P}$ is a profile and $i \in V$.

Definition

A pointed profile $(\mathbf{P}, i)$ **witnesses manipulability for** resolute voting method f if and only if there is a profile $\mathbf{P}'$ differing from $\mathbf{P}$ only in i's ranking such that:

> the winner in $f(\mathbf{P}')$ is preferred by $\mathbf{P}_i$ to the winner in $f(\mathbf{P})$.

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Intuitive idea: if we regard $\mathbf{P}_i$ as i's sincere preference, then by the lights of i's sincere preference, the winner would be better if i were to submit an insincere preference $\mathbf{P}'_i$.

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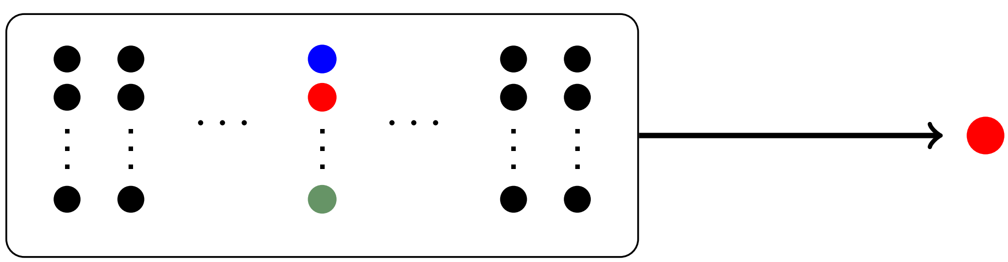
Intuitive idea: if we regard $\mathbf{P}_i$ as i's sincere preference, then by the lights of i's sincere preference, the winner would be better if i were to submit an insincere preference $\mathbf{P}'_i$.
Then f is **manipulable** if there is a $(\mathbf{P}, i)$ witnessing manipulation.

Manipulation



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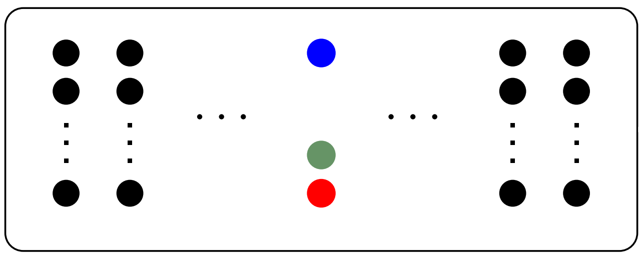

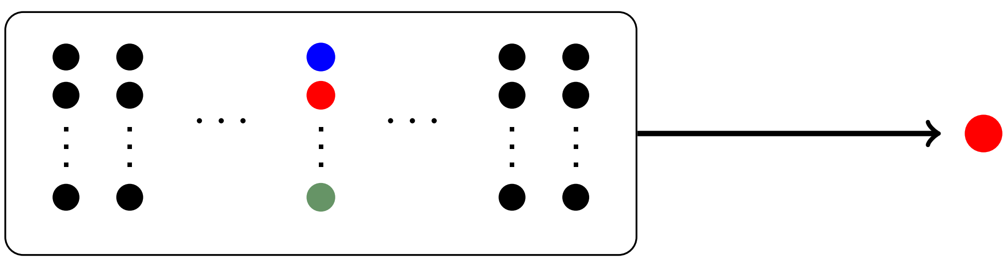

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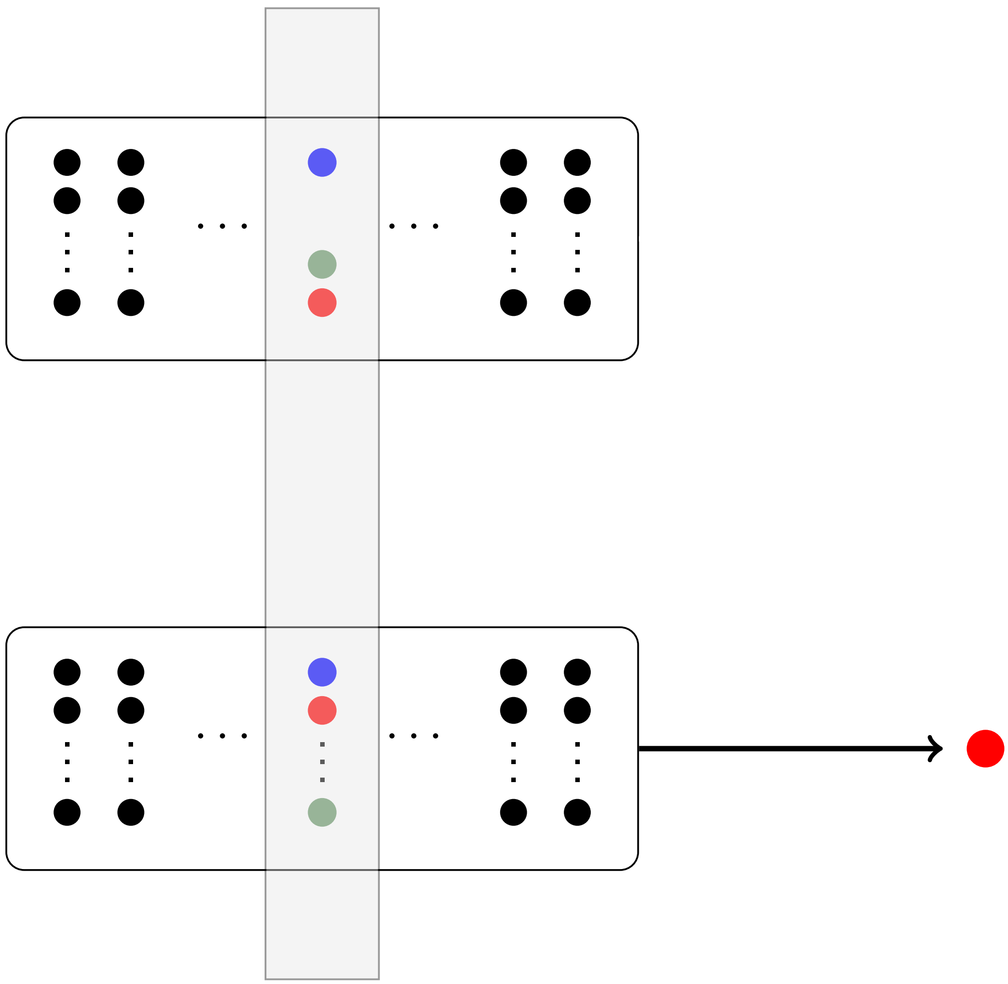

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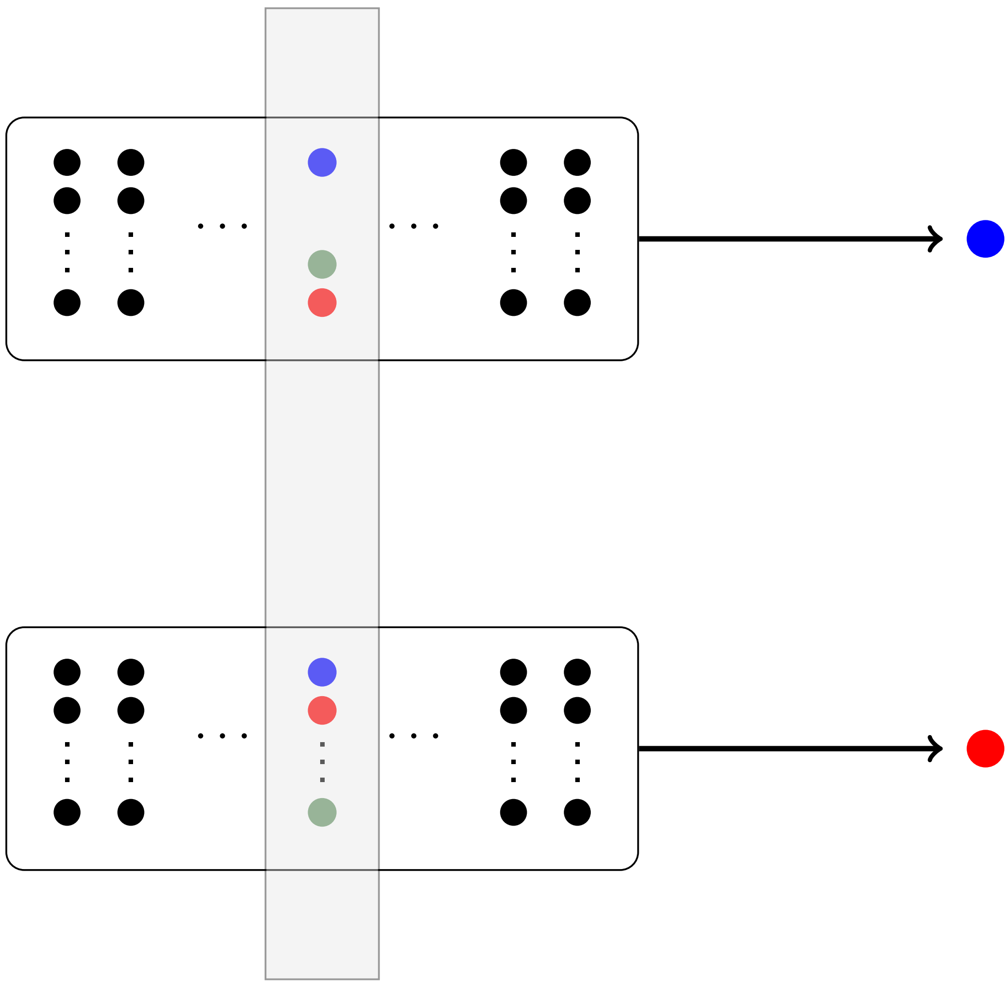

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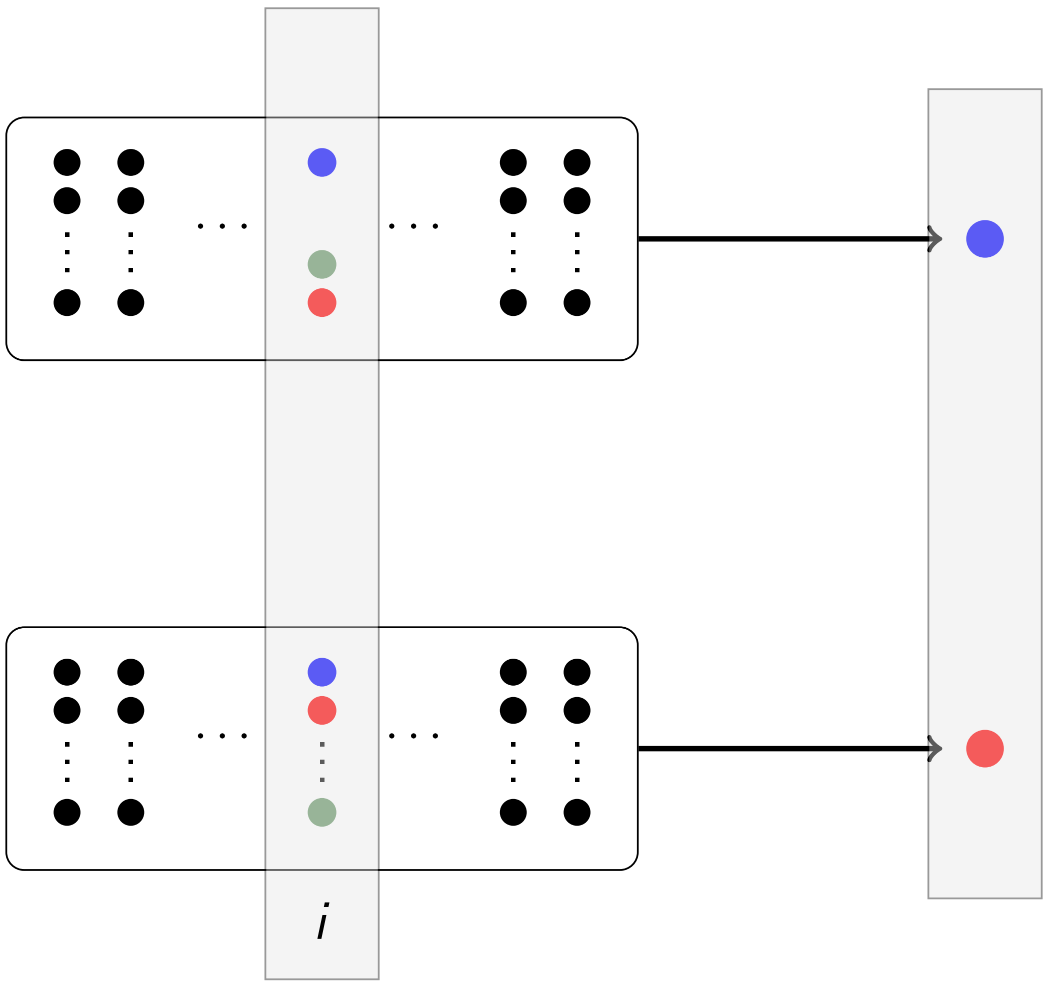

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a	b	a	a	c

b wins

Since **voter 1 prefers** b **to** a in the top left profile $\mathbf{P}$, we have that $(\mathbf{P}, 1)$ witnesses manipulability for `Hare`.

The Gibbard-Satterthwaite Theorem

Theorem (Gibbard 1973, Satterthwaite 1975)

If f is a resolute voting method for $|C| \geq 3$ that is

1. non-dictatorial ($\neg \exists i \in V \, \forall \mathbf{P} \in \mathrm{Profiles} : f(\mathbf{P}) = \{\max(\mathbf{P}_i)\}$),
2. onto ($\forall c \in C \, \exists \mathbf{P} \in \mathrm{Profiles} : f(\mathbf{P}) = \{c\}$)

then f is manipulable.

A. Gibbard. *Manipulation of voting schemes: A general result*. Econometrica, 41(4): 587-601, 1973.

M. A. Satterthwaite. *Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions*. Journal of Economic Theory, 10(2): 187-217, 1975.

Manipulation of Irresolute Rules

To talk about manipulation for irresolute rules, we need a notion of lifting a preference over alternatives to sets of alternatives.

Definition

Let $\mathbf{P}$ be a profile, $i \in V$, and $X, Y \subseteq C$. We define the following dominance notions (where $a\mathbf{R}_i b$ iff $a\mathbf{P}_i b$ or $a = b$):

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1. weak dominance:
   1.1 $X \geq^{weak}_{\mathbf{P}_i} Y$ if and only if $\forall x \in X\ \forall y \in Y$: $x\mathbf{R}_i y$;
   1.2 $X >^{weak}_{\mathbf{P}_i} Y$ if and only if $X \geq^{weak}_{\mathbf{P}_i} Y$ and $\exists x \in X\ \exists y \in Y$: $x\mathbf{P}_i y$.

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1. weak dominance:
    1.1 $X \geq_{\mathbf{P}_i}^{weak} Y$ if and only if $\forall x \in X\ \forall y \in Y$: $x\mathbf{R}_i y$;
    1.2 $X >_{\mathbf{P}_i}^{weak} Y$ if and only if $X \geq_{\mathbf{P}_i}^{weak} Y$ and $\exists x \in X\ \exists y \in Y$: $x\mathbf{P}_i y$.
2. optimistic dominance:
    2.1 $X >_{\mathbf{P}_i}^{Opt} Y$ if and only if $\max(X, \mathbf{P}_i)\ \mathbf{P}_i\ \max(Y, \mathbf{P}_i)$.

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1. weak dominance:
   1.1 $X \geq^{weak}_{\mathbf{P}_i} Y$ if and only if $\forall x \in X\ \forall y \in Y$: $x\mathbf{R}_i y$;
   1.2 $X >^{weak}_{\mathbf{P}_i} Y$ if and only if $X \geq^{weak}_{\mathbf{P}_i} Y$ and $\exists x \in X\ \exists y \in Y$: $x\mathbf{P}_i y$.
2. optimistic dominance:
   2.1 $X >^{Opt}_{\mathbf{P}_i} Y$ if and only if $\max(X, \mathbf{P}_i)\ \mathbf{P}_i\ \max(Y, \mathbf{P}_i)$.
3. pessimistic dominance:
   3.1 $X >^{Pes}_{\mathbf{P}_i} Y$ if and only if $\min(X, \mathbf{P}_i)\ \mathbf{P}_i\ \min(Y, \mathbf{P}_i)$.

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We say that $(\mathbf{P}, i)$ **witnesses Δ-dominance manipulability for** f iff there is a profile $\mathbf{P}'$ differing from $\mathbf{P}$ only in i's ranking s.th.:

$$f(\mathbf{P}') >^{\Delta}_{\mathbf{P}_i} f(\mathbf{P}).$$

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The Duggan-Schwartz Theorem

Theorem (Duggan-Schwartz 2000)

If f is a voting method for $|C| \geq 3$ that

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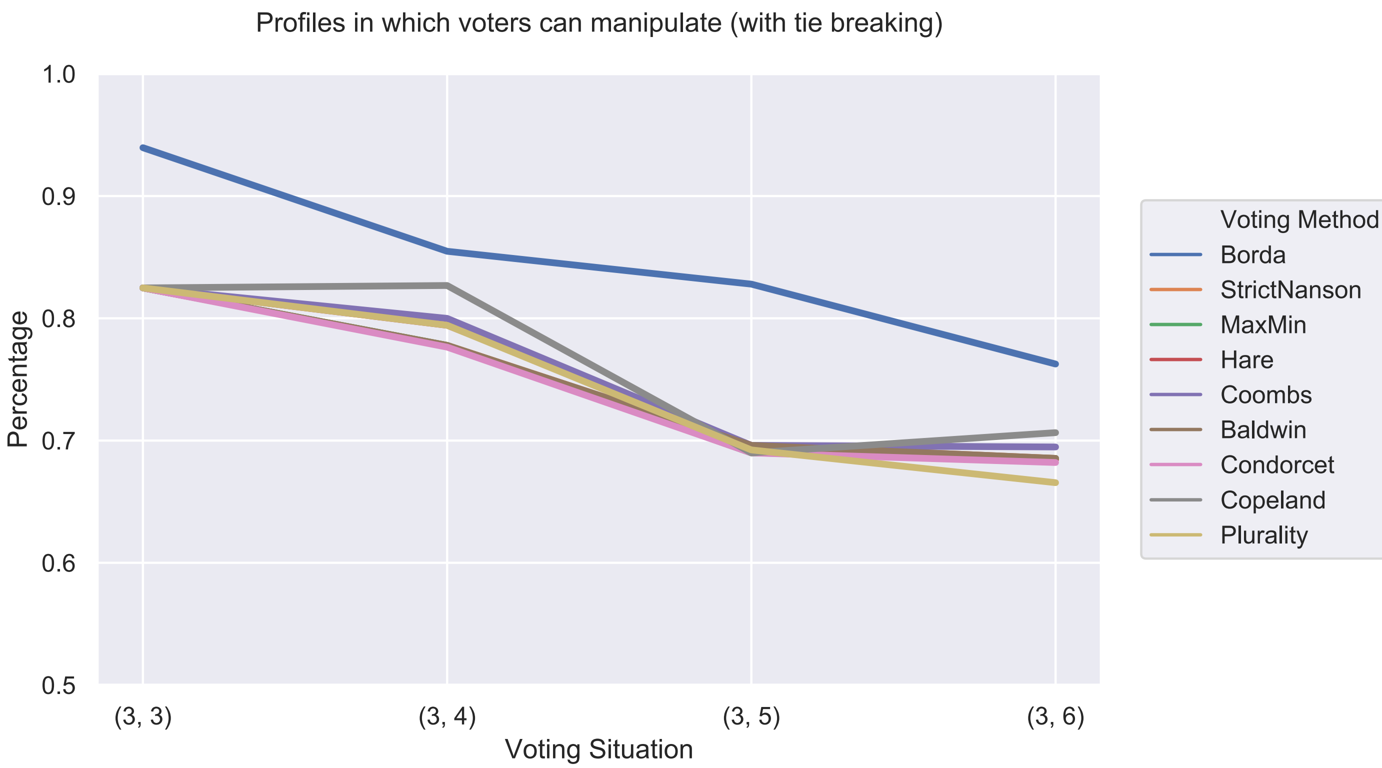
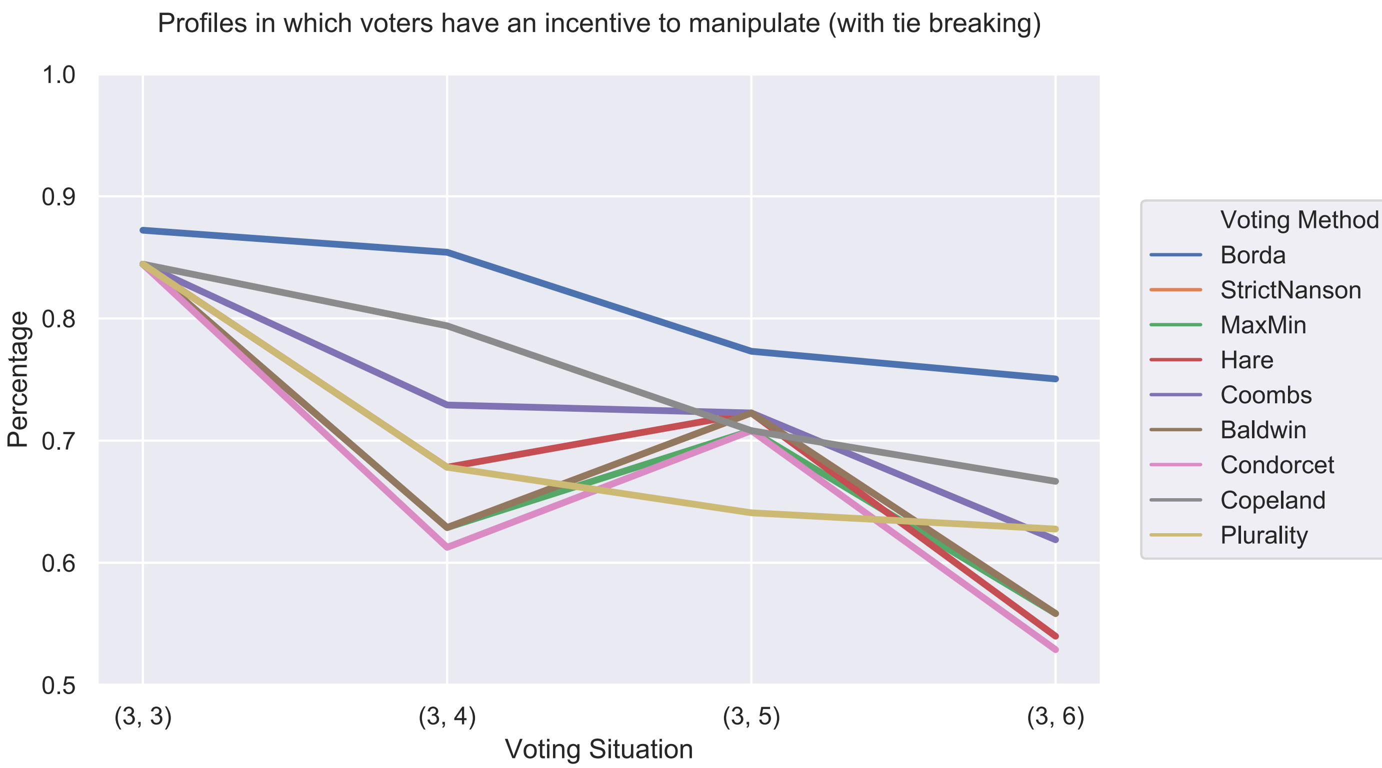
1. has no nominator ($\neg \exists i \in V\, \forall \mathbf{P} \in \mathrm{Profiles} : \max(\mathbf{P}_i) \in f(\mathbf{P})$) and
2. is non-imposed ($\forall c \in C\, \exists \mathbf{P} \in \mathrm{Profiles} : f(\mathbf{P}) = \{c\}$),

then f is manipulable by an optimist or manipulable by a pessimist.

J. Duggan and T. Schwartz. *Strategic manipulability without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized*. Social Choice and Welfare, 17: 85-93, 2000.

S. Nitzan. *The vulnerability of point-voting schemes to preference variation and strategic manipulation*. Public Choice 47 (1985), 349 - 370, 1985.


Profiles in which voters can manipulate (with tie breaking)
1.0
0.9
0.8
0.7
0.6
0.5
Percentage
(3, 3)
(3, 4)
(3, 5)
(3, 6)
Voting Situation
Voting Method
Borda
StrictNanson
MaxMin
Hare
Coombs
Baldwin
Condorcet
Copeland
Plurality



Profiles in which voters have an incentive to manipulate (with tie breaking)
1.0
0.9
0.8
0.7
0.6
0.5
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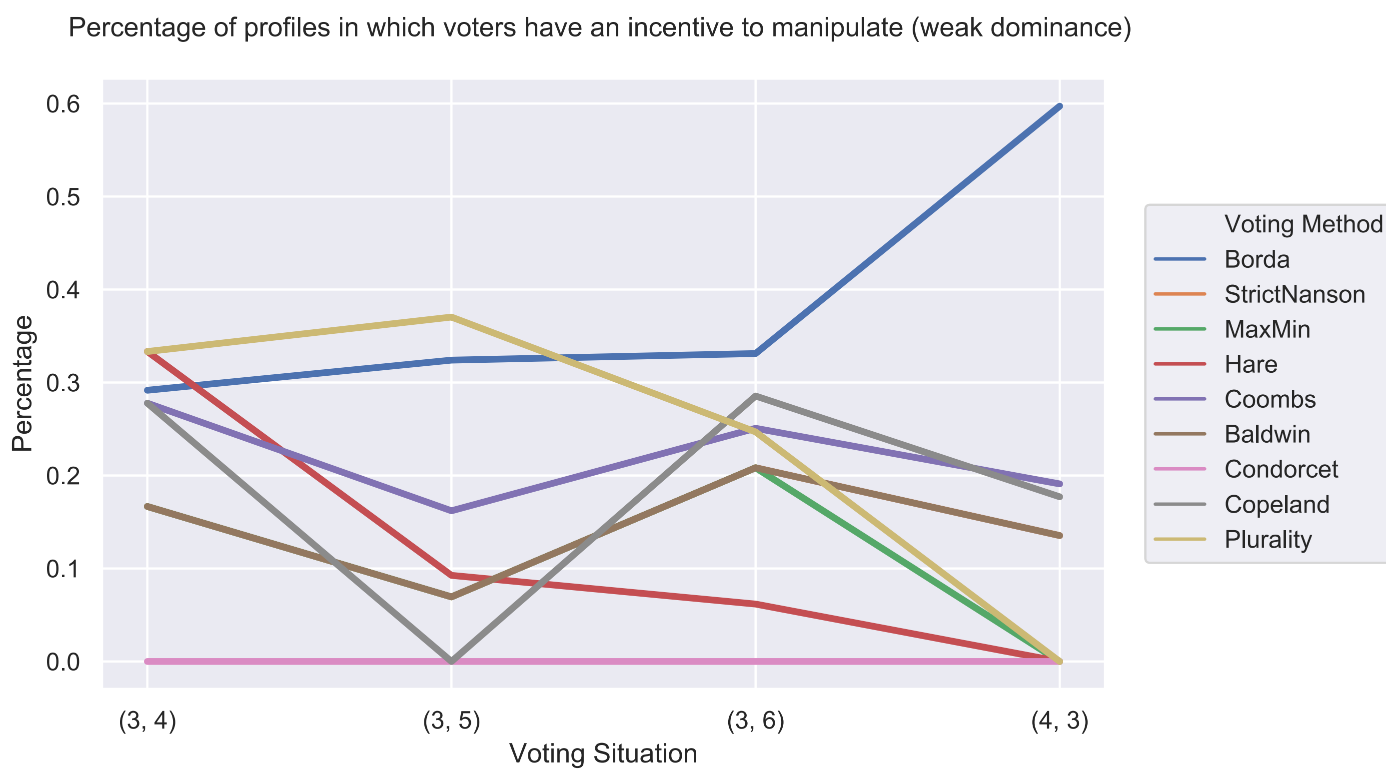
Plurality

What is the percentage of profiles that has some voter with an incentive to manipulate using weak dominance/optimist/pessimist dominance?


Percentage of profiles in which voters have an incentive to manipulate (weak dominance)
Percentage
0.0
0.1
0.2
0.3
0.4
0.5
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(3, 4)
(3, 5)
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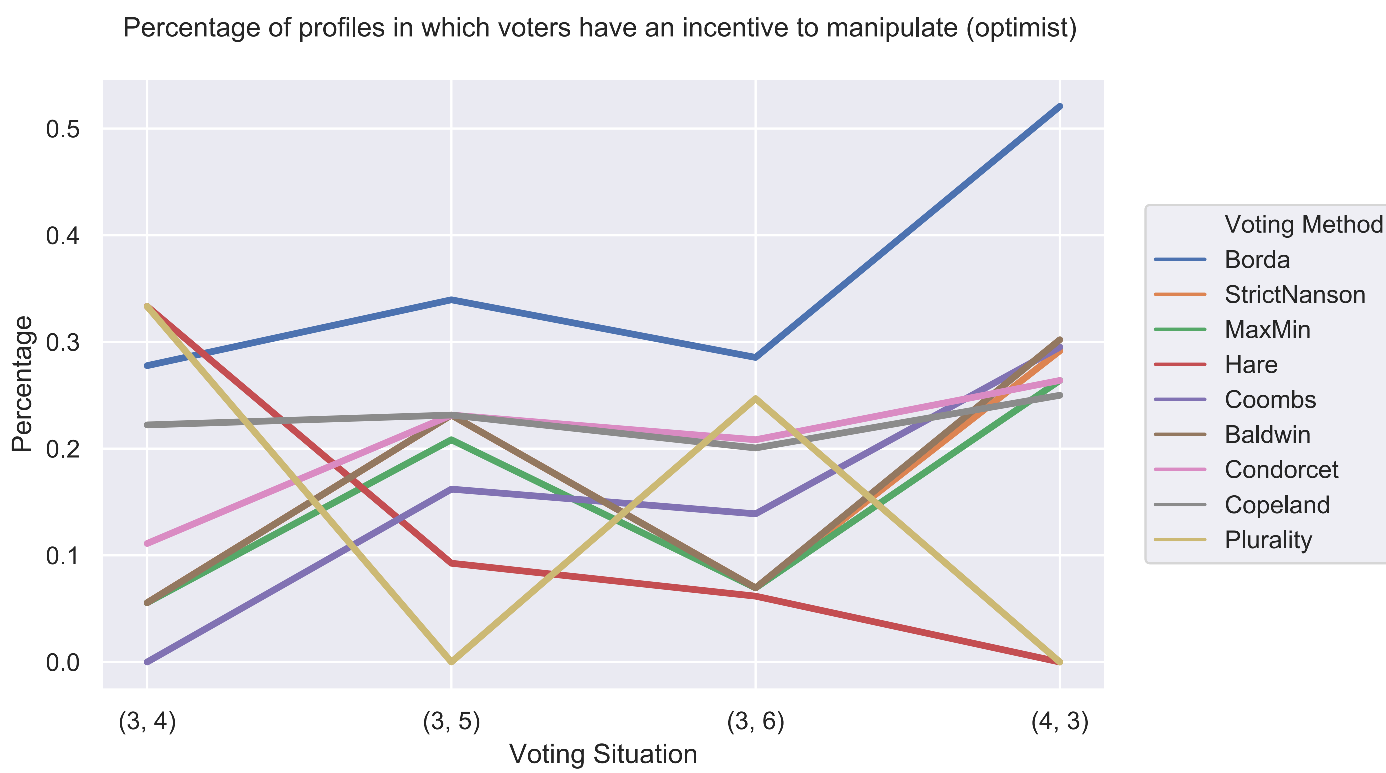
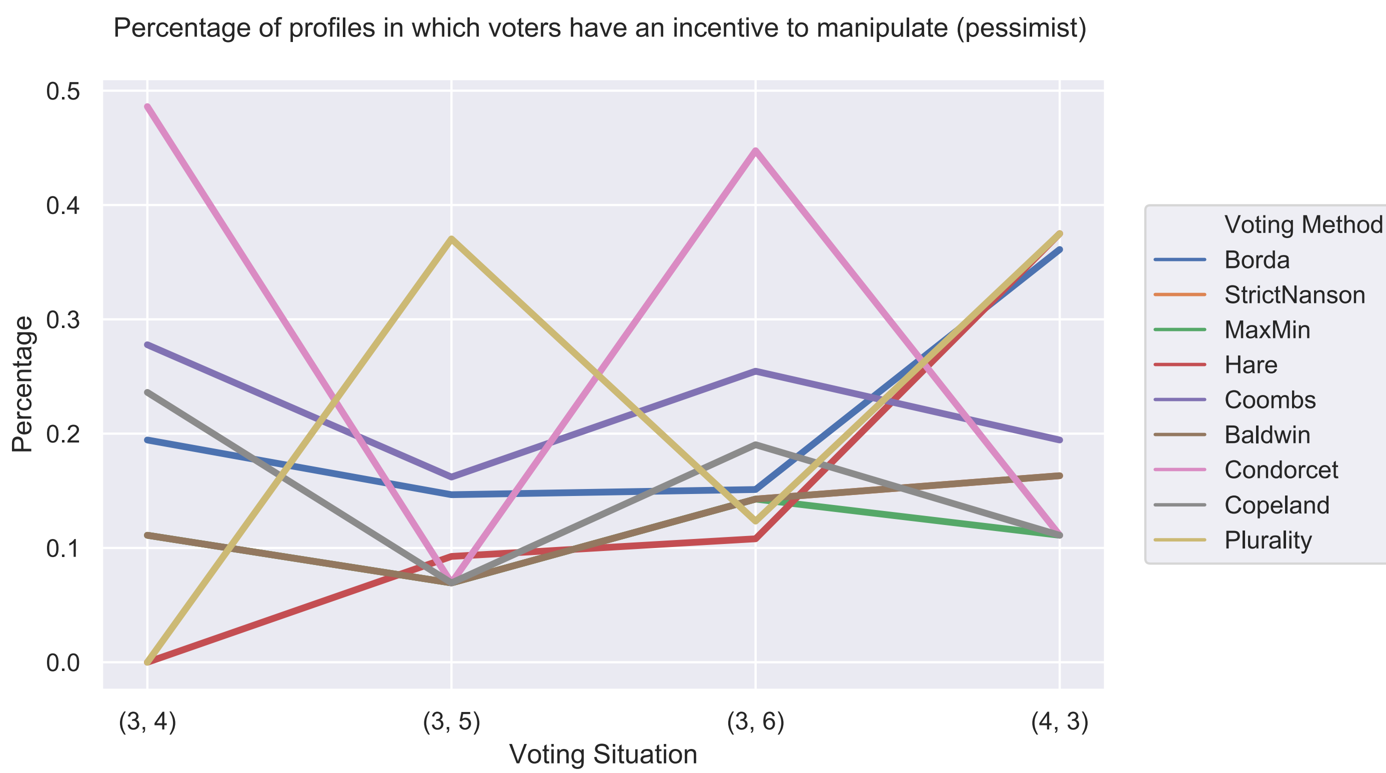
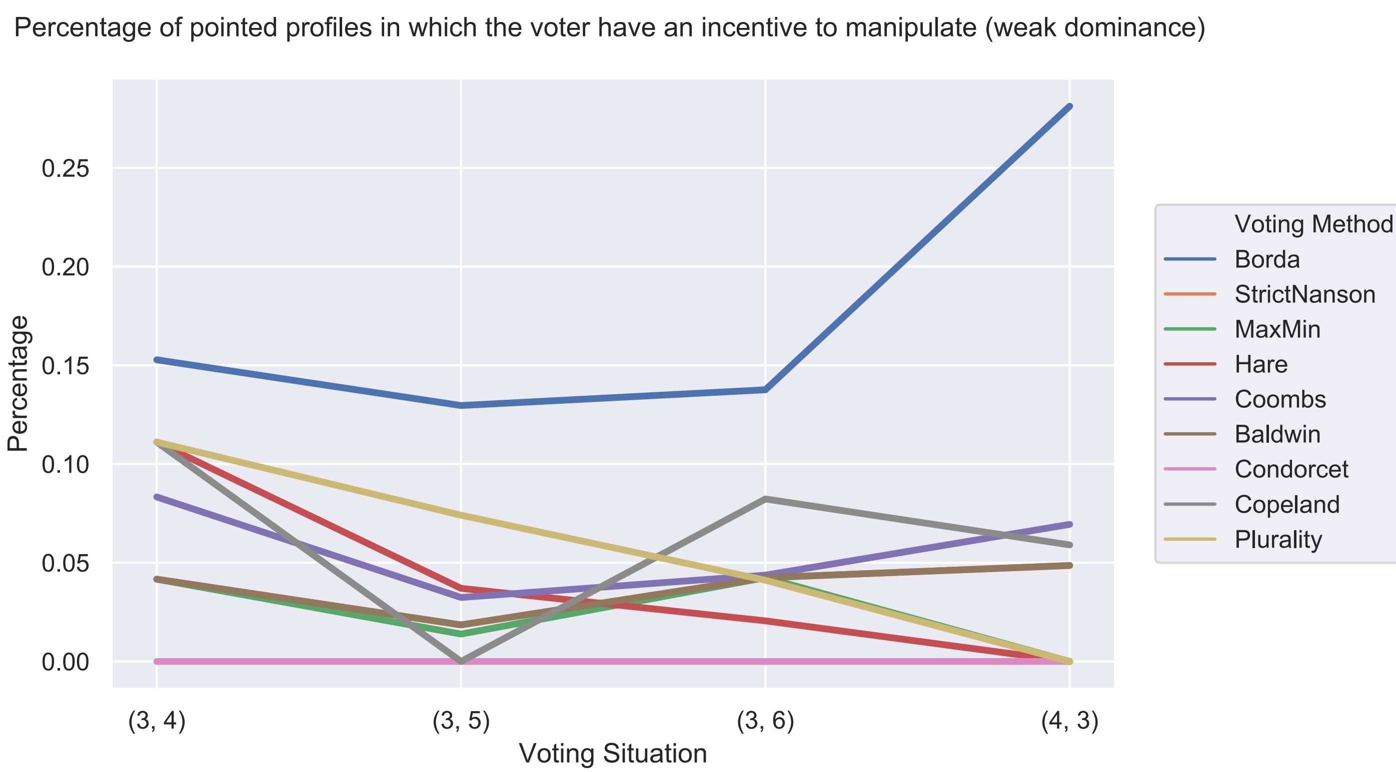
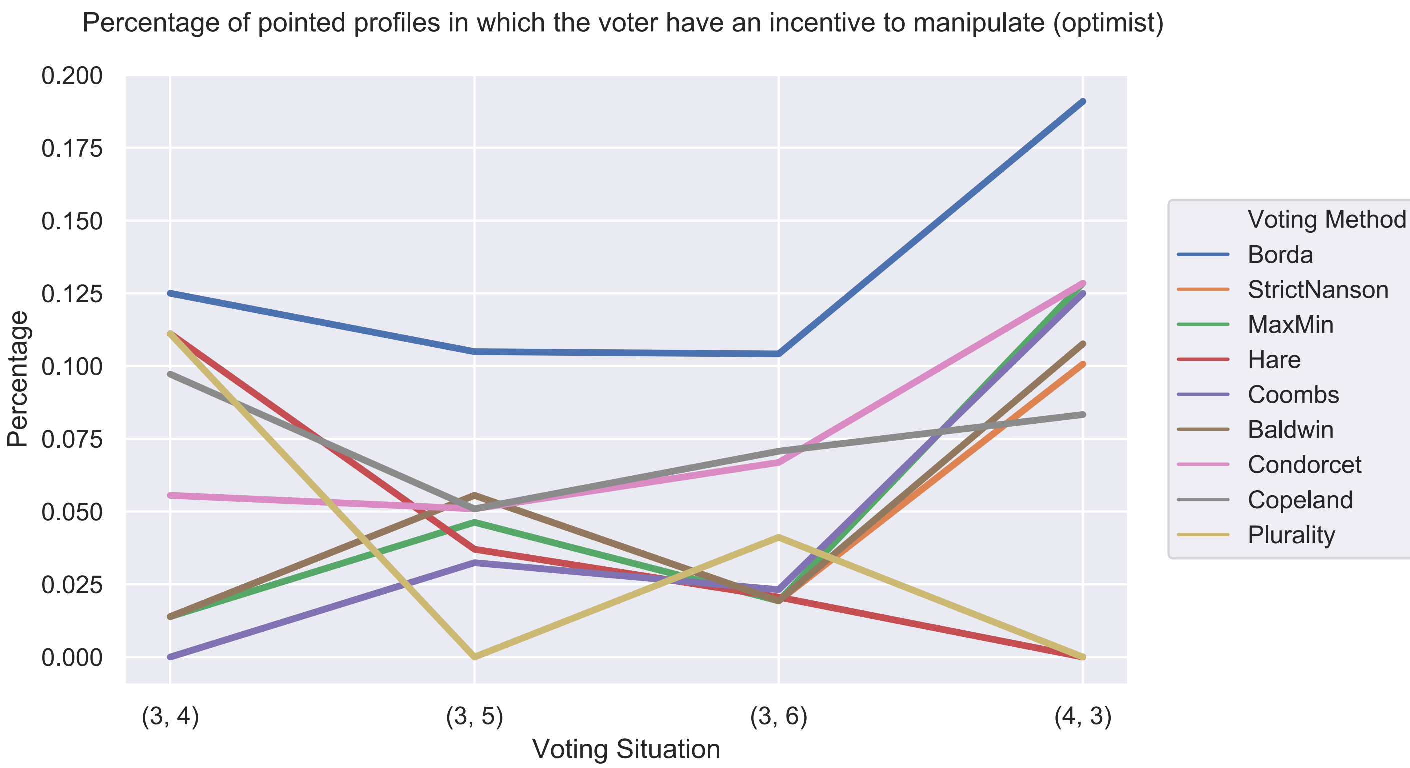
Coombs
Baldwin
Condorcet
Copeland
Plurality


Percentage of profiles in which voters have an incentive to manipulate (optimist)
Percentage
0.0
0.1
0.2
0.3
0.4
0.5
(3, 4)
(3, 5)
(3, 6)
(4, 3)
Voting Situation
Voting Method
Borda
StrictNanson
MaxMin
Hare
Coombs
Baldwin
Condorcet
Copeland
Plurality

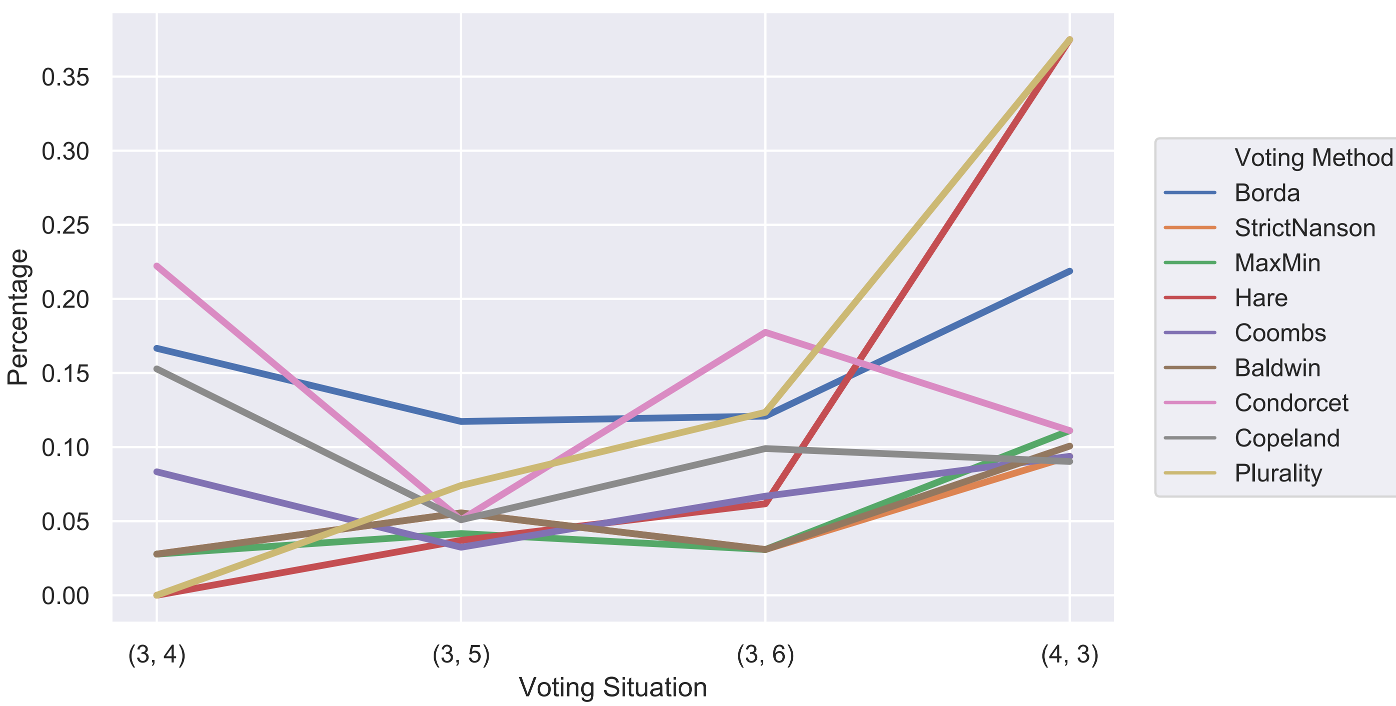

Percentage of profiles in which voters have an incentive to manipulate (pessimist)
Percentage
0.0
0.1
0.2
0.3
0.4
0.5
(3, 4)
(3, 5)
(3, 6)
(4, 3)
Voting Situation
Voting Method
Borda
StrictNanson
MaxMin
Hare
Coombs
Baldwin
Condorcet
Copeland
Plurality

What is the percentage of pointed profiles in which the voter has an incentive to manipulate using weak dominance/optimist/pessimist dominance?


Percentage of pointed profiles in which the voter have an incentive to manipulate (weak dominance)
Percentage
0.25
0.20
0.15
0.10
0.05
0.00
(3, 4)
(3, 5)
(3, 6)
(4, 3)
Voting Situation
Voting Method
Borda
StrictNanson
MaxMin
Hare
Coombs
Baldwin
Condorcet
Copeland
Plurality


Percentage of pointed profiles in which the voter have an incentive to manipulate (optimist)
Percentage
0.200
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
(3, 4)
(3, 5)
(3, 6)
(4, 3)
Voting Situation
Voting Method
Borda
StrictNanson
MaxMin
Hare
Coombs
Baldwin
Condorcet
Copeland
Plurality

Percentage of pointed profiles in which the voter have an incentive to manipulate (pessimist)



Anonymous and Neutral Equivalence Classes

Ö. Eğecioğlu and A. E. Giritligil. *The Impartial, Anonymous, and Neutral Culture Model: A Probability Model for Sampling Public Preference Structures*. Journal of Mathematical Sociology, 37, 203 – 222, 2013.

Y. Veselova. *The difference between manipulability indices in the IC and IANC models*. Social Choice and Welfare 46 (2016), 609 - 638, 2016.

E. Mossel and M. Z. Rácz. *A quantitative Gibbard-Satterthwaite theorem without neutrality*. Combinatorica 35, 3 (2015), 317 - 387, 2015.

Closeness of f and g:

$C(f, g) = \frac{1}{(m!)^n} \sum_{\mathbf{P} \in L(X)^V} [\![f(\mathbf{P}) = g(\mathbf{P})]\!]$

f is ϵ-**bad** if there is some voting rule g_f such that g_f is a dictator (or only elects one of two candidates) and $C(g_f, f) \geq 1 - \epsilon$.

Manipulation power of voter i:

$M_i(f) = \frac{1}{(m!)^n} \sum_{\mathbf{P} \in L(X)^V} [\![\exists P' \text{ s.t. } f(\mathbf{P}_{-i}, P') \; \mathbf{P}_i \; f(\mathbf{P})]\!]$

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Theorem (Mossel and Rácz). For any $\epsilon \geq 0$ and any resolute voting rule, either f is ϵ-bad, or $\sum_{i \in V} M_i(f) > p(\frac{1}{n}, \frac{1}{m}, \epsilon)$ where p is somoe polynomial function with positive coefficients.

Assumptions of this Literature

The standard notion of manipulation assumes the following:

Assumptions of this Literature

The standard notion of manipulation assumes the following:

1. the strategizing voter i knows how the other voters will vote or have voted (or at least i believes this).
   - E.g., in a faculty meeting where faculty reveal their preferences over the job candidates sequentially, i is last in line.
2. the voter assumes that the other voters will not change their vote (e.g., group vs. individual manipulation).
3. the strategizing voter i knows which voting method will determine the winner(s).
4. voters have unlimited computational power to determine an alternative vote.

Barriers to Strategic Voting?

One potential barrier against strategic voting, investigated mostly in the AI literature, is the computational complexity of determining a profitable strategic vote for a given voting method.

P. Faliszewski and A. Procaccia. *AI's War on Manipulation: Are We Winning?*. AI Magazine, 2010.

V. Conitzer and T. Walsh. *Barriers to Manipulation in Voting*. Handbook of Computational Social Choice, 2016.

Summary

- Which assumptions of the theorem or “crucial”? (non-resolute social choice functions, weak orders, partial orders, domain restrictions: single-peaked domains, etc.)
- Can randomization help?
- What’s so *bad* about manipulation?
- How “hard” is it to manipulate an election?
- When will a voter misrepresent her preferences? What do the voters need to *know* (believe) about the other voters’ preferences?
- What is the effect of polling information on an election outcome?

Safe Manipulation

10	15	14	2
a	c	e	e
b	e	b	d
c	b	c	a
d	a	a	c
e	d	d	b

8	2	15	14	2
b	a	c	e	e
a	b	e	b	d
d	c	b	c	a
c	d	a	a	c
e	e	d	d	b

- e is the Borda winner on the left
- 8 voters in the first group change their ranking
- b is the Borda winner on the right.

Safe Manipulation

10	15	14	2
a	c	e	e
b	e	b	d
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b	a	c	e	e
a	b	e	b	d
d	c	b	c	a
c	d	a	a	c
e	e	d	d	b

- e is the Borda winner on the left
- 8 voters in the first group change their ranking
- b is the Borda winner on the right.
- *However if 2-6 of the voters in the first group submit this ranking, then e is the winner.*

Safe Manipulation

A manipulation P is **safe** for i if (i) there is some set of voters that submit P leading to an outcome that is preferable for i and (ii) no set of voters using P lead to a strictly worse outcome for i.

A. Slinko and S. White. *Is it ever safe to vote strategically?*. Social Choice and Welfare, 43(2), pp. 403 - 427, 2014.

1	2	3	4	5
c	a	b	c	a
b	c	a	b	c
a	b	c	a	b

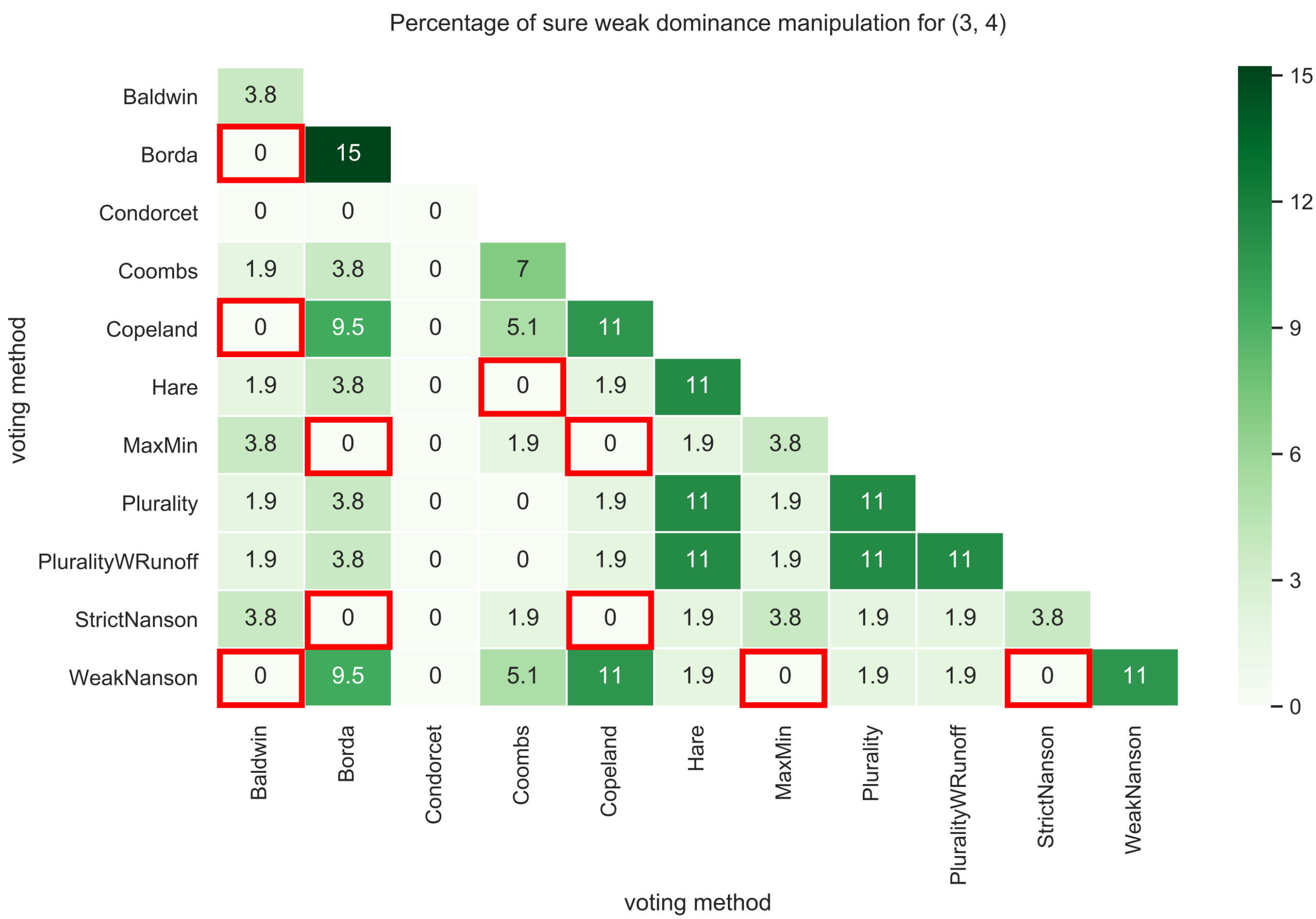
1	2	3	4	5
b	a	b	c	a
a	c	a	b	c
c	b	c	a	b

- $\texttt{Hare}(\mathbf{P}) = \{a\}$ (c has fewest 1st place votes so gets eliminated in round 1; then b has two 1st place votes vs. a's three) and $\texttt{Borda}(\mathbf{P}) = \{c\}$.
- $\texttt{Hare}(\mathbf{P}') = \{b\}$ (c has fewest 1st place votes so gets eliminated in round 1; then b has three 1st place votes vs. a's two) and $\texttt{Borda}(\mathbf{P}') = \{a\}$.
- Since $\{b\} >^{weak}_{\mathbf{P}} \{a\}$, 1 has an incentive to manipulate with Hare. But not with Borda, since $\{c\} >^{weak}_{\mathbf{P}_1} \{a\}$.

So if there is uncertainty about which of Borda or Hare will be used, then the manipulation is **not safe**!

W. Holliday and E. Pacuit. *Strategic Voting Under Uncertainty About the Voting Method*. Proceedings of TARK, `http://eptcs.web.cse.unsw.edu.au/paper.cgi?TARK2019:44.pdf`.

Sure Weak Dominance Manipulation



F. Brandt. *Rolling the Dice: Recent Results in Probabilistic Social Choice*. Handbook of Computational Social Choice, 2016.

Let $V = \{1, \ldots, n\}$ be a set of voters, X a set of m alternatives.

The set of all lotteries over X is:

$$\Delta(X) = \{p \in \mathbb{R}^X \ : \ p(x) \geq 0 \text{ for all } x \in X \text{ and } \sum_{x \in X} p(x) = 1\}$$

A probabilistic social choice function (PSCF) is a map $F : O(X)^V \to \wp(\Delta(A)) \setminus \emptyset$ such that for all $\mathbf{P}$, $F(\mathbf{P})$ is a convex set of lotteries.

Anonymity and neutrality can be defined as usual.

Random (Serial) Dictator

Random dictatorship: A voter is picked uniformly at random and this voter's most-preferred alternative is selected. Thus, the probabilities assigned by RD are directly proportional to the number of agents who top-rank a given alternative (or, in other words, the alternative's plurality score).

Random serial dictatorship (RSD): RSD selects a permutation of the agents uniformly at random and then sequentially allows agents in the order of the permutation to narrow down the set of alternatives to their most preferred of the remaining ones.

- $Borda_{max}$ yields all lotteries that randomize over alternatives with maximal Borda score.
- $Borda_{pro}$ assigns probabilities to the alternatives that are proportional to their Borda scores.

Theorem (Gibbard, 1977). *RD* is the only anonymous, strongly *SD-strategyproof*, and *ex post efficient* PSCF when preferences are strict.

- Every misreported preference relation of an agent will result in a lottery q such that $p \succeq^{SD} q$, where

  $p \succeq^{SD} q$ iff for all x, $\sum_{\{y \mid yPx\}} p(y) \geq \sum_{\{y \mid yPx\}} q(y)$

  ($p \succeq^{SD} q$ iff, for every vNM utility function compatible with P, the expected utility for p is at least as large as that for q)

- A PSCF is *ex post efficient* if it puts probability 0 on all Pareto dominated alternatives

Theorem (Brandl et al., 2016). There is no anonymous, neutral, *SD*-efficient, and *SD*-strategyproof PSCF when $m, n \geq 4$.

SD-efficient means that it does not return a lottery that is *SD*-dominated.

F. Brandl, F. Brandt, and C. Geist. *Proving the incompatibility of efficiency and strategyproofness via SMT solving*. Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI), pgs. 116 - 122. AAAI Press, 2016.

Theorem (Brandl et al., 2016). There is no anonymous, neutral, *SD*-efficient, and *SD*-strategyproof PSCF when $m, n \geq 4$.

SD-efficient means that it does not return a lottery that is *SD*-dominated.

Note: "This impossibility was obtained with the help of a computer and the proof is long and tedious to verify for humans. It has been verified by the interactive theorem prover Isabelle/HOL"

F. Brandl, F. Brandt, and C. Geist. *Proving the incompatibility of efficiency and strategyproofness via SMT solving*. Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI), pgs. 116 - 122. AAAI Press, 2016.

M. Nunez and M. Pivato. *Truth-Revealing Voting Rules for Large Populations*. to appear in Games and Economic Behaviour, 2019.

With probability $1 - q$, select the winner using the deterministic Borda rule. With probability q, we use the following random device instead:

1. First randomly choose one of the voters i and any pair of alternatives a and b.
2. If n prefers a to b, then select a. Otherwise, select b.

M. Nunez and M. Pivato. *Truth-Revealing Voting Rules for Large Populations*. to appear in Games and Economic Behaviour, 2019.

When confronted with the random device, i has a unique dominant strategy: reveal her true ordinal preferences.

On the other hand, under the deterministic Borda rule, she will have an incentive to misrepresent her true preferences *only when her vote is pivotal*.

But if the probability of such a pivotal event is small enough relative to q, then the expected utility gain from misrepresenting her preferences becomes negligible in comparison with the expected utility loss of misrepresenting her preferences when confronted with the random device.

Much more to talk about...

- Impossibility theorems
- Probabilistic social choice
- Voting by grading (Approval Voting, Majority Judgement, Score Voting)
- Alternative voting methods (quadratic voting, liquid democracy)
- Characterization results/Voting methods as statistical estimators
- Strategic voting
- Behavioral social choice

Approval Voting: Each voter selects a subset of candidates. The candidate with the most "approvals" wins the election.

S. Brams and P. Fishburn. *Approval Voting*. Birkhauser, 1983.

J.-F. Laslier and M. R. Sanver (eds.). *Handbook of Approval Voting*. Studies in Social Choice and Welfare, 2010.

Under Approval Voting (AV), voters are asked to select the candidates that the voter *approves*.

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The two pieces of information are related, but not derivable from each other

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Under ranking voting procedures (such as Borda Count), voters are asked to (linearly) rank the candidates.

The two pieces of information are related, but not derivable from each other

Approving of a candidate is not (necessarily) the same as simply ranking the candidate first.

Why Approval Voting?

www.electology.org/approval-voting

S. Brams and P. Fishburn. *Going from Theory to Practice: The Mixed Success of Approval Voting*. Handbook of Approval Voting, pgs. 19-37, 2010.

Approval Voting is more flexible

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$, vote-for-2 elects $\{D\}$

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$, vote-for-2 elects $\{D\}$, vote-for-3 elects $\{A, B\}$.

Approval Voting is more flexible

AV may elect the Condorcet winner

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
$(\{A\}, \{B\}, \{C, A\})$ elects A under AV.

Possible Failure of Unanimity

# voters	1	1	1
	A	C	D
	B	A	A
	C	B	B
	D	D	C

Possible Failure of Unanimity

# voters	1	1	1
	A	C	D
	B	A	A
	C	B	B
	D	D	C

Approval Winners: A, B

Indeterminate or Responsive?

# voters	6	5	4
	A	B	C
	C	C	B
	B	A	A

Plurality winner: A, Borda and Condorcet winner: C.

Indeterminate or Responsive?

# voters	6	5	4
	A	B	C
	C	C	B
	B	A	A

Plurality winner: A, Borda and Condorcet winner: C.
Any combination of A, B and C can be an AV winner (or AV winners).

Generalizing Approval Voting

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Ask the voters to provide both a linear ranking of the candidates and the candidates that they approve.

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Ask the voters to provide both a linear ranking of the candidates and the candidates that they approve.

Make the ballots more expressive: Dis&Approval voting, RangeVoting, Majority Judgement

Grading

In many group decision situations, people use measures or grades from a **common language of evaluation** to evaluate candidates or alternatives:

- in figure skating, diving and gymnastics competitions;
- in piano, flute and orchestra competitions;
- in classifying wines at wine competitions;
- in ranking university students;
- in classifying hotels and restaurants, e.g., the Michelin $*$

Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)

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Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)
- How should we *aggregate* the grades? (e.g., Average or Median)
- Should there be a "no opinion" option?

Voting by Grading: Examples

Approval Voting: voters can assign a single grade "approve" to the candidates

Dis&Approval Voting: voters can approve or disapprove of the candidates

Majority Judgement, **Score Voting**: voters can assign any grade from a fixed set of grades to the candidates

Strong Paradox of Grading Systems

Grades: $\{0, 1, 2, 3\}$
Candidates: $\{A, B, C\}$
3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Grades: $\{0, 1, 2, 3\}$
Candidates: $\{A, B, C\}$
3 Voters

# voters	1	1	1	Avg
A	3	2	0	5/3
B	0	3	1	4/3
C	0	3	1	4/3

Average Grade Winner: A

Grades: $\{0, 1, 2, 3\}$
Candidates: $\{A, B, C\}$
3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$B \succ A$$

Grades: $\{0, 1, 2, 3\}$
Candidates: $\{A, B, C\}$
3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$C \sim B \succ A$

Grades: $\{0, 1, 2, 3\}$
Candidates: $\{A, B, C\}$
3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$C \sim B \succ A$$

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

Superior Grade Winners: C, B

Grades: $\{0, 1, 2, 3, 4, 5\}$
Candidates: $\{A, B, C\}$
5 Voters

# voters	1	4	Avg
A	5	0	5/5
B	0	1	4/5
C	0	1	4/5

Average Grade Winner: A

Superior Grade Winner: B, C

To conclude, we have identified a paradox of grading systems, which is not just a mirror of the well-known differences that crop up in aggregating votes under ranking systems. Unlike these systems, for which there is no accepted way of reconciling which candidate to choose when, for example, the Hare, Borda and Condorcet winners differ, AV provides a solution when the AG and SG winners differ.

Theorem (Brams and Potthoff). When there are two grades, the AG and SG winners are identical.

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