Preference Elicitation & Recommendation

Applied Mechanism Design for Social Good — CMSC828M
21 April, 2020

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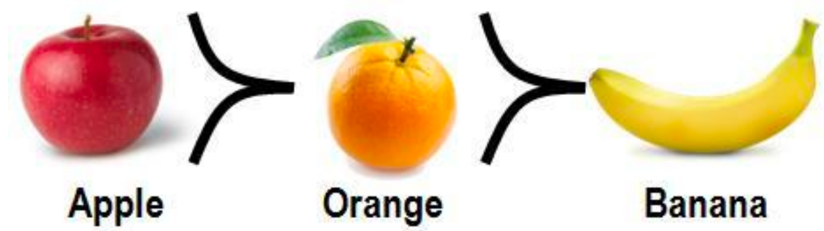
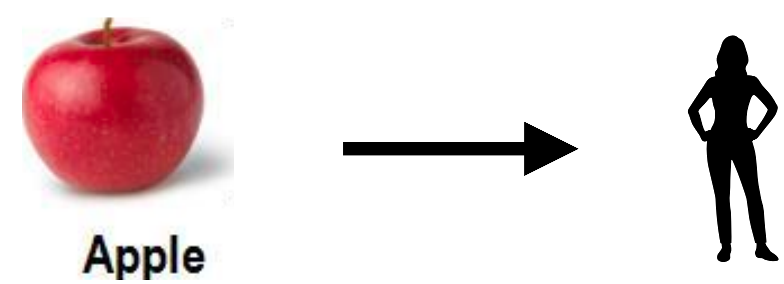
Outline

- Application: Learning an Objective Function

- Preference Elicitation



- Recommendation Under Uncertainty



- Elicitation + Recommendation

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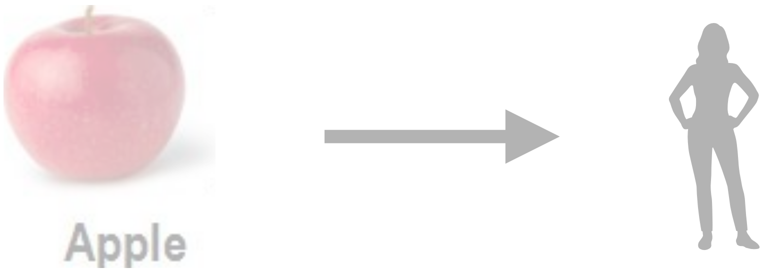

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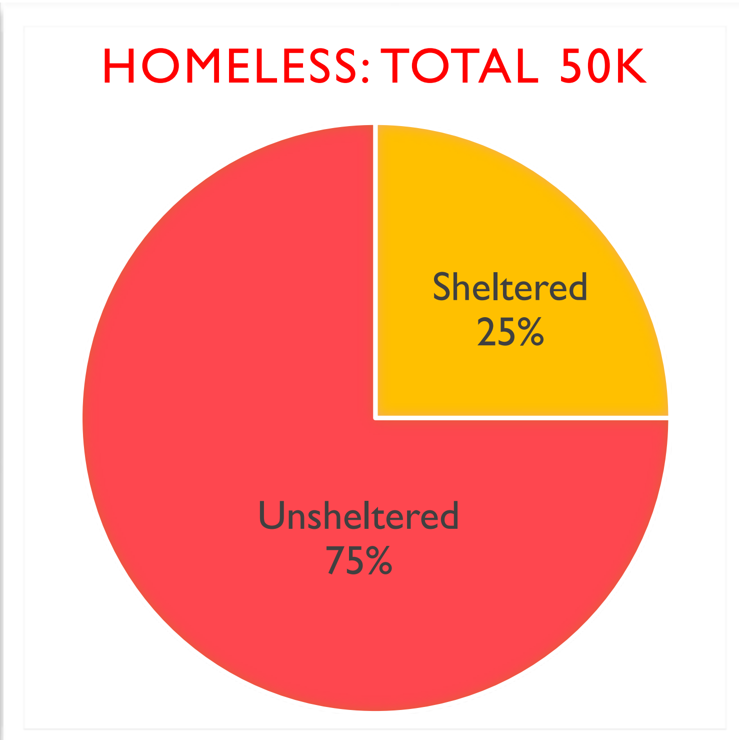
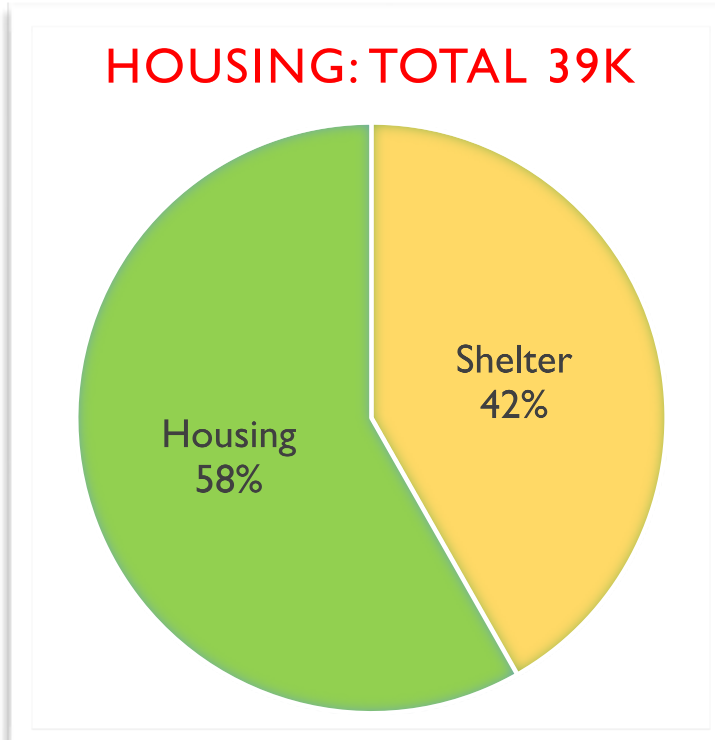
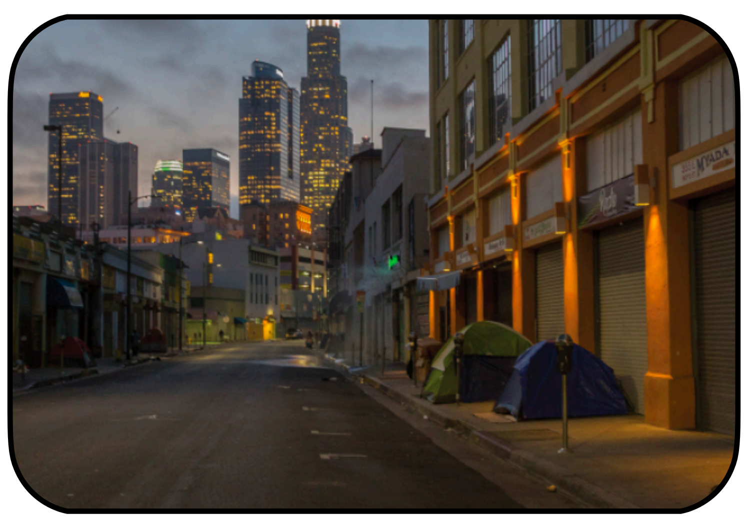
Application: Housing the Homeless





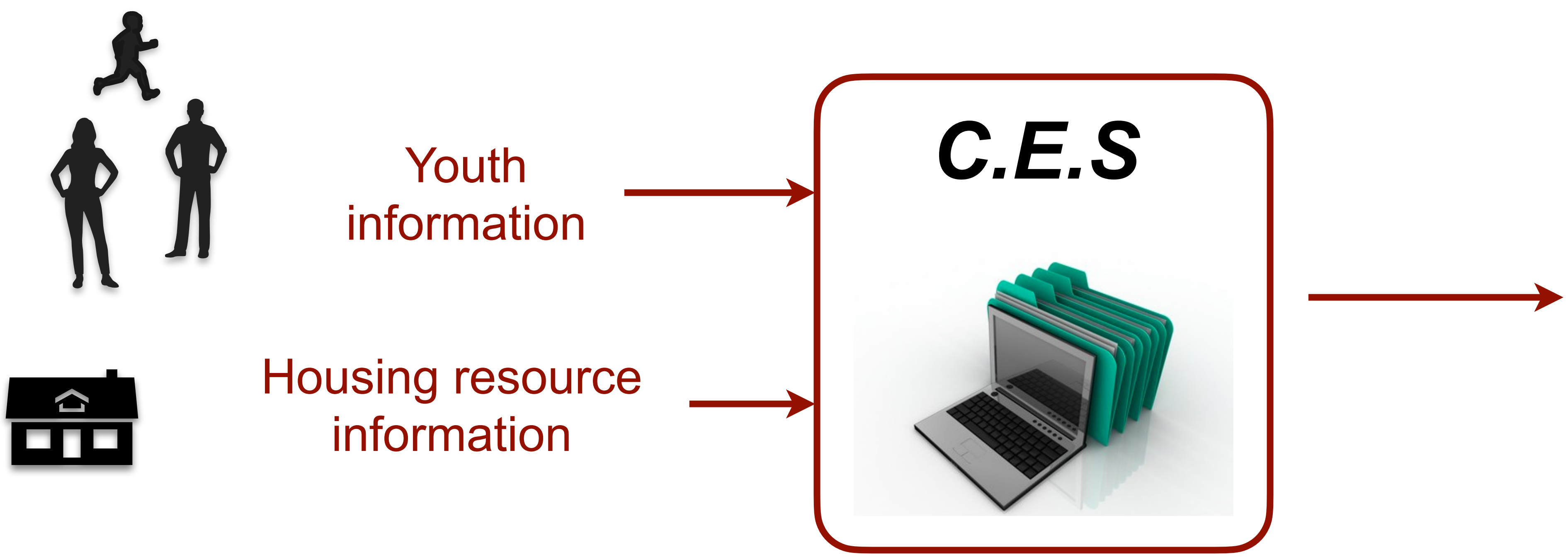
LAHSA
LOS ANGELES HOMELESS SERVICES AUTHORITY



Housing assistance is a **scarce resource**

Application: Housing the Homeless

Coordinated Entry Systems

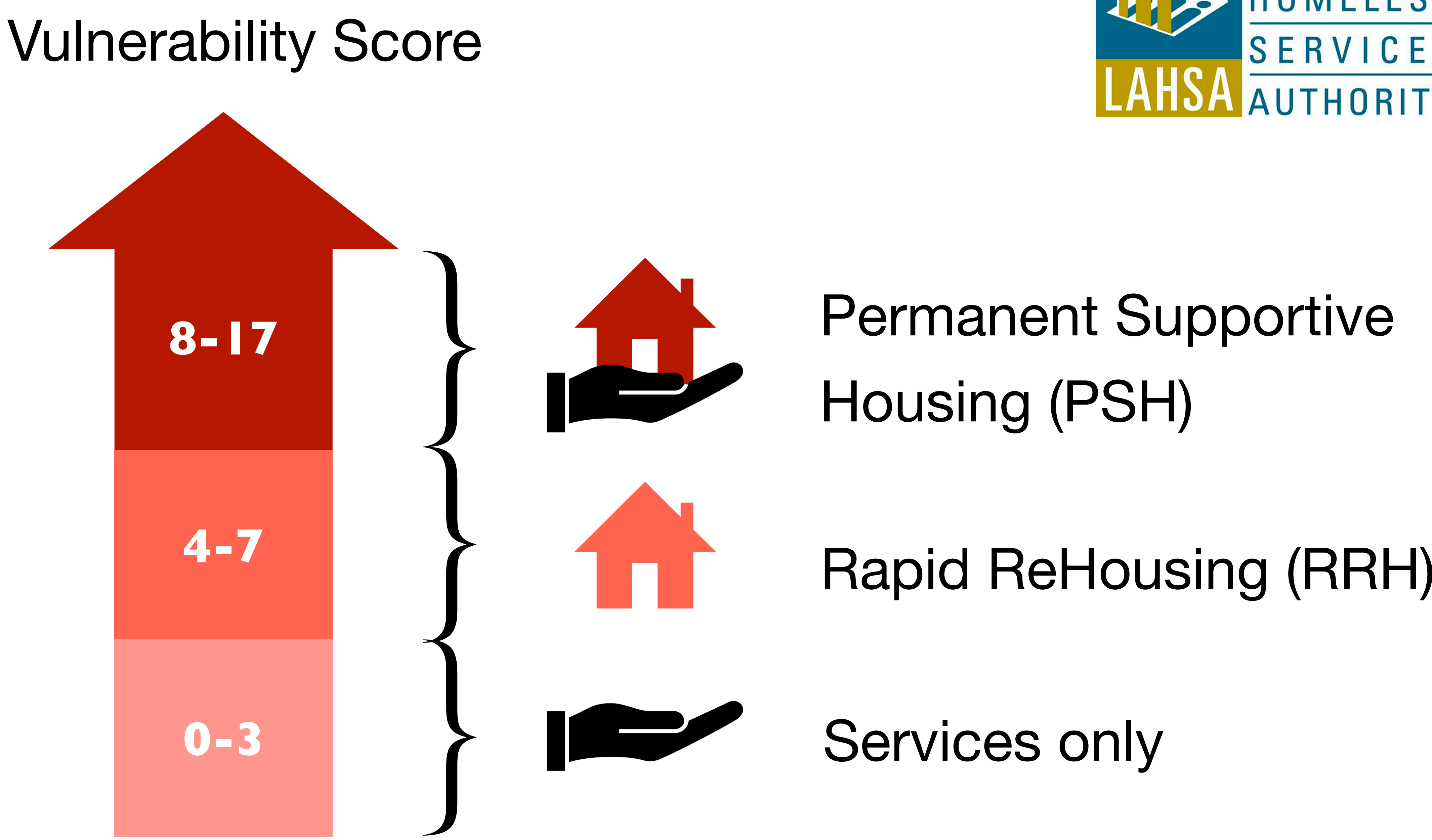


1	John
2	Maria
3	Alexander
4	Lea
…	…

*: https://www.hud.gov/sites/dfiles/SPM/documents/HUDSTRATEGICPLAN2018-2022.pdf

Application: Housing the Homeless

Current Policy



Big Picture

What is the current state of affairs? → ***What do stakeholders want?*** → ***Can we do better?***

USC Center for Artificial Intelligence in Society

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Linking vulnerability scores to outcomes
[Rice et al. (2018) *Cityscape*]

Analyzing waiting times for youth
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What do stakeholders want?

Today

Can we do better?

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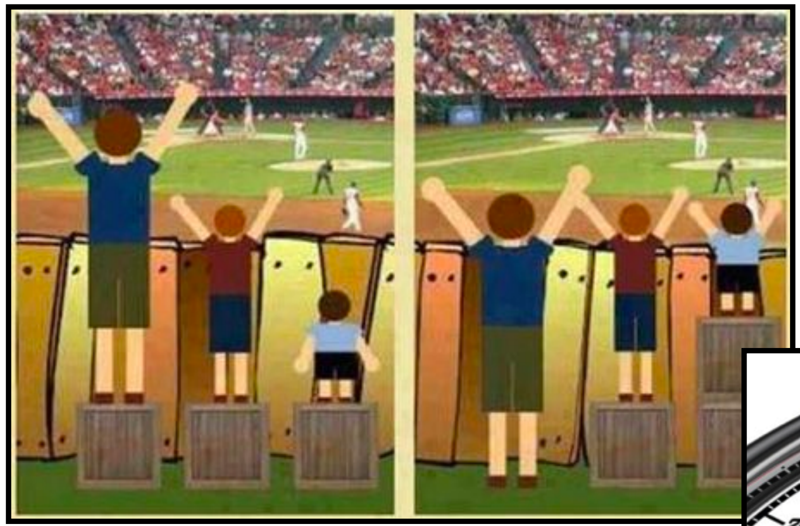
Motivation: Housing the Homeless

Policy Priorities

- What are the policy priorities?
- How important is each?



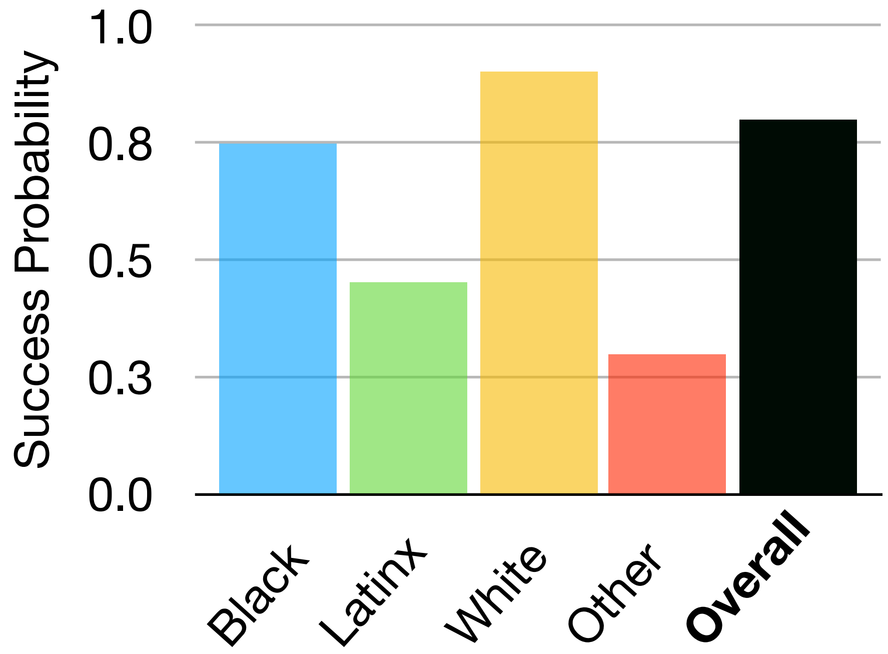
Fairness

Efficiency

Interpretability

Balancing Policy Priorities

Efficiency vs. Equity



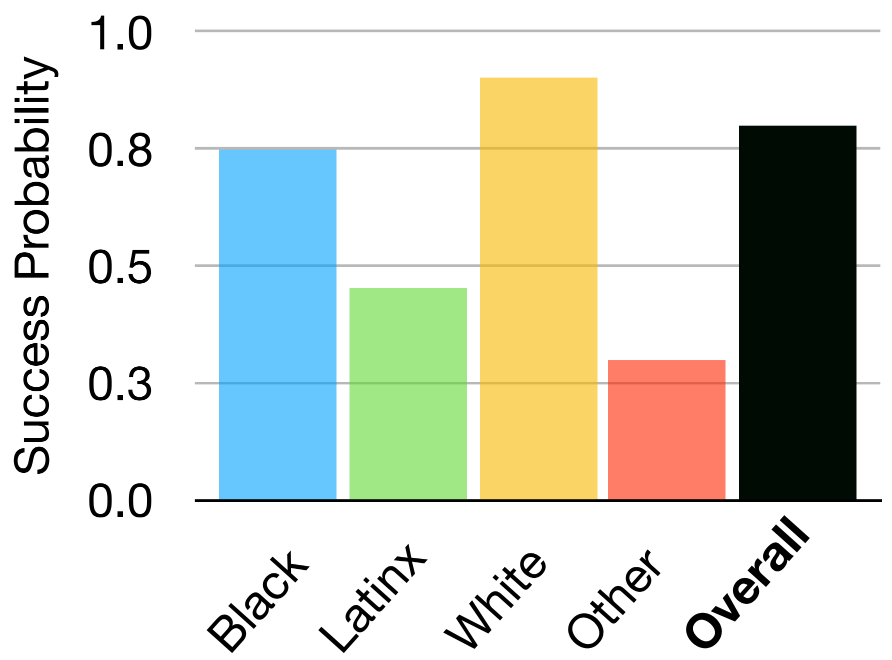
Average success probability

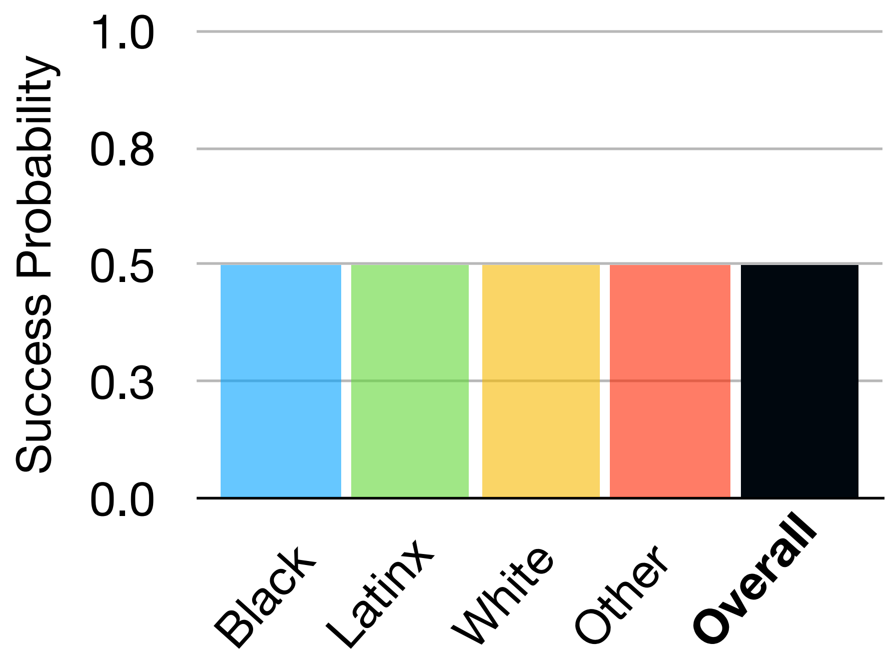
i.e.

probability of exiting homelessness

Balancing Policy Priorities

Efficiency vs. Equity





Which policy is better?

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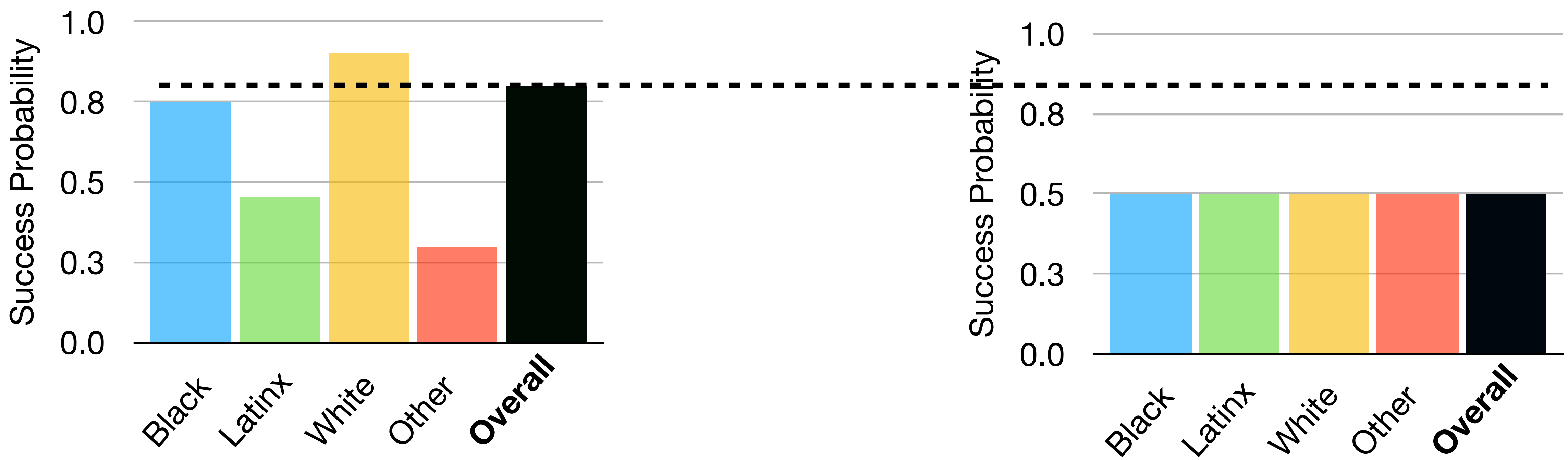

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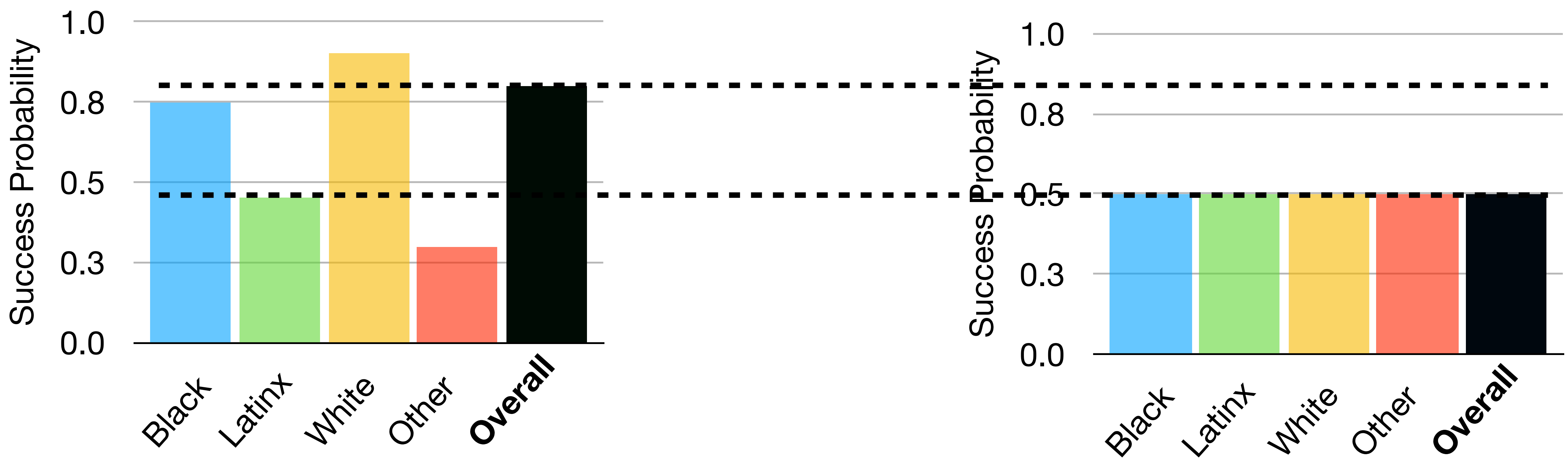

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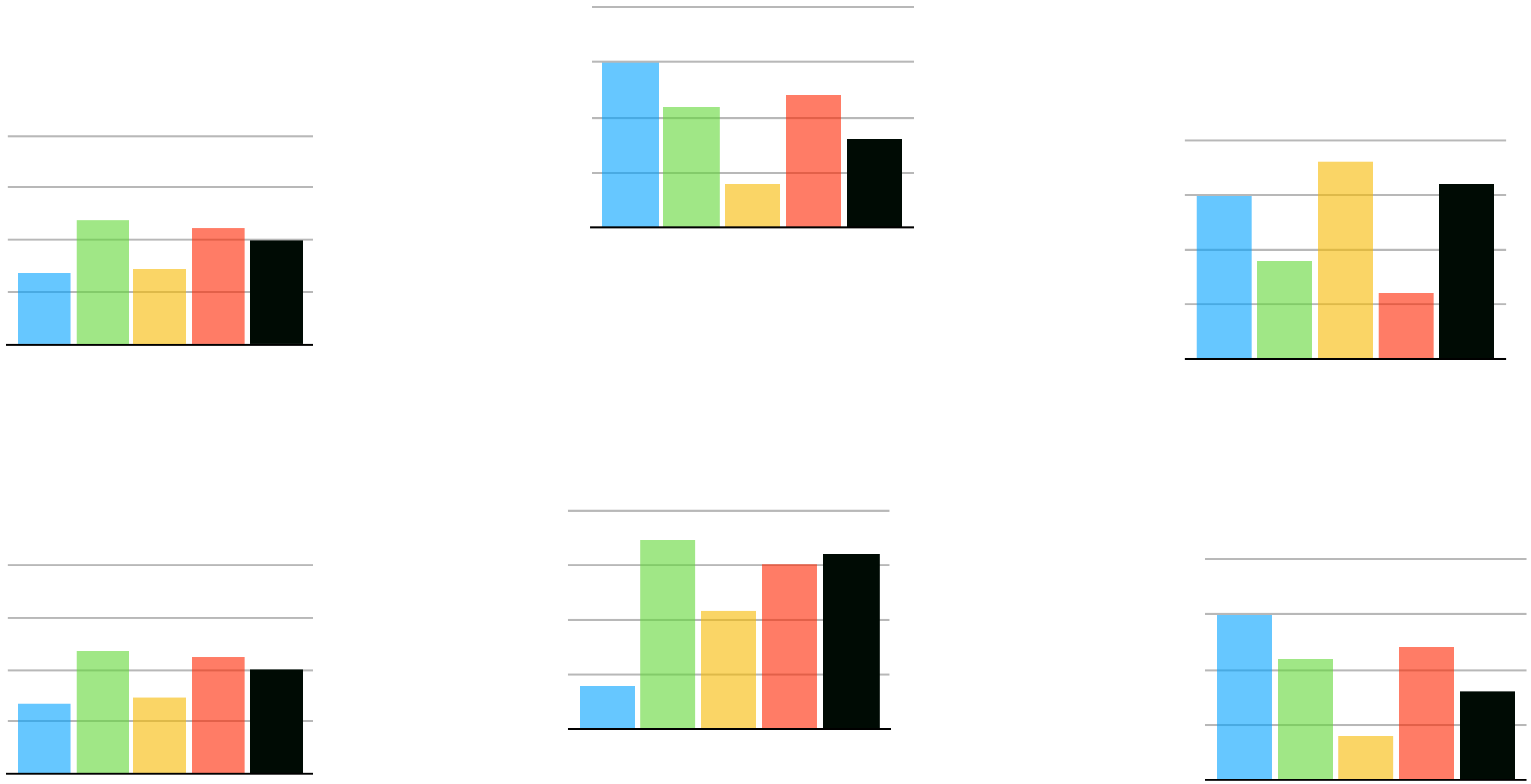

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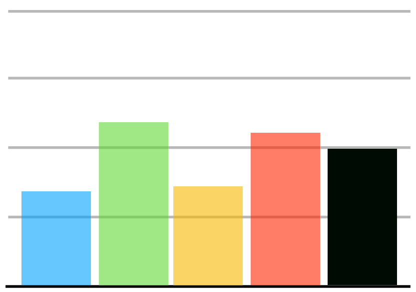

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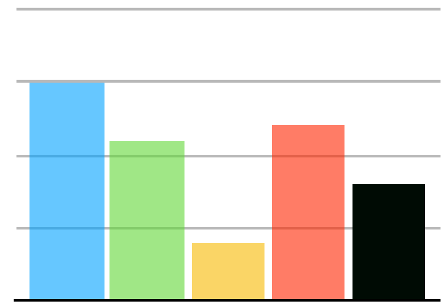

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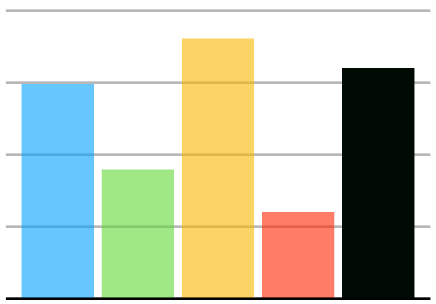

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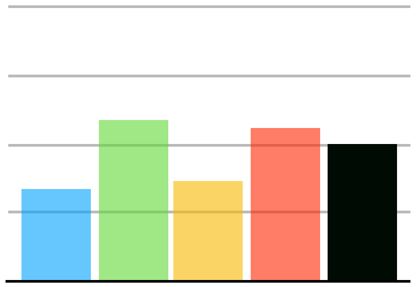

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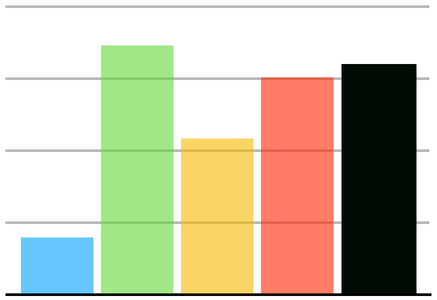

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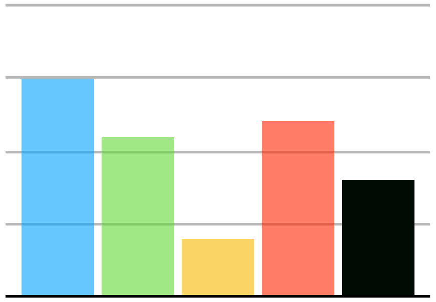

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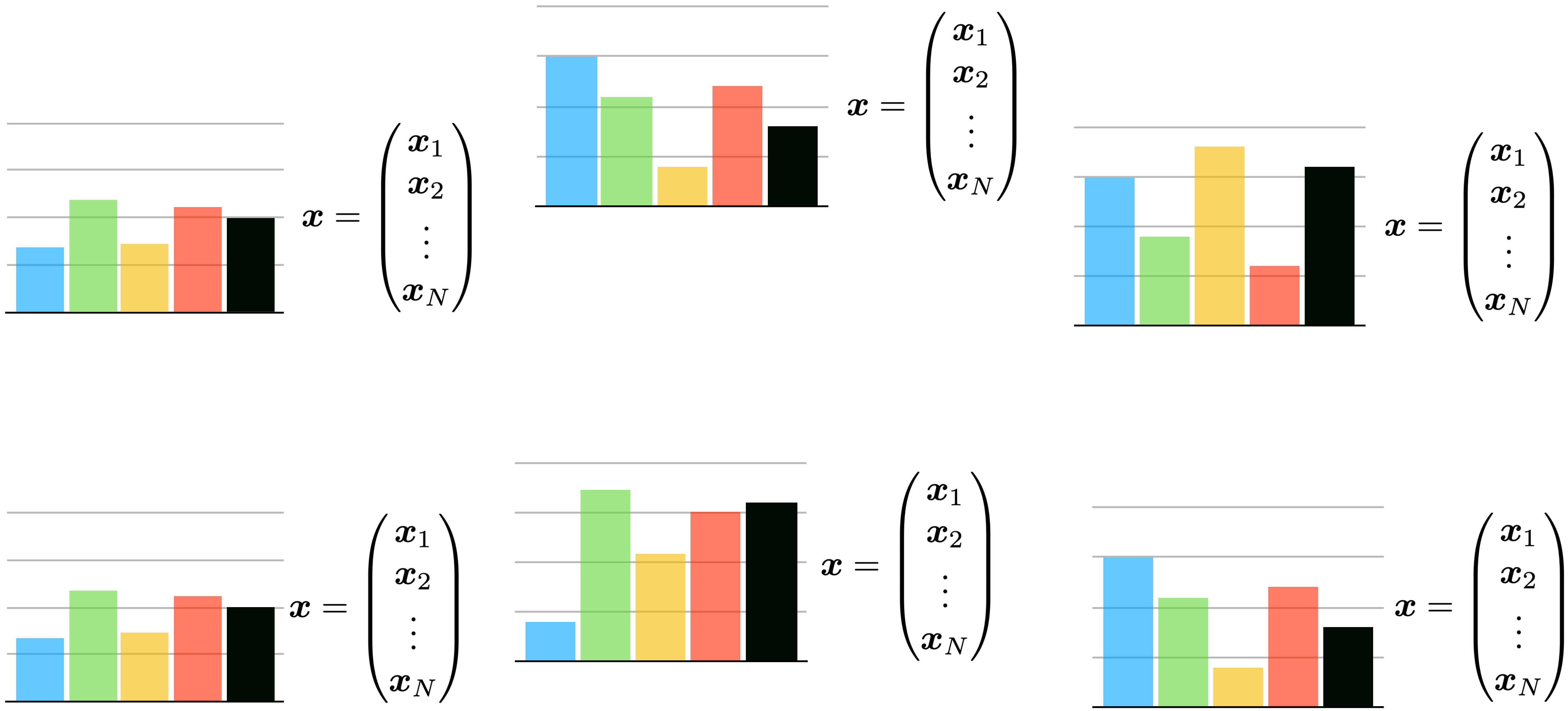


Which policy is **best**? $\boldsymbol{x} \in \mathcal{X}$

Elicitation & Recommendation

Preference Elicitation

What does the {agent | customer | user} want?

Marketing

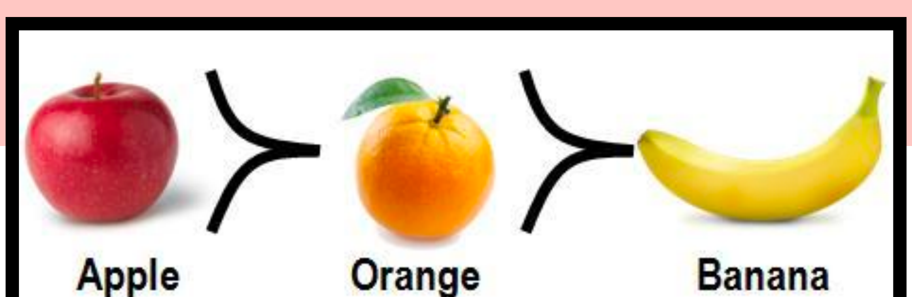
What products does the customer like?

Medicine

What procedures do patients like?

Public Policy

Which projects to citizens like?



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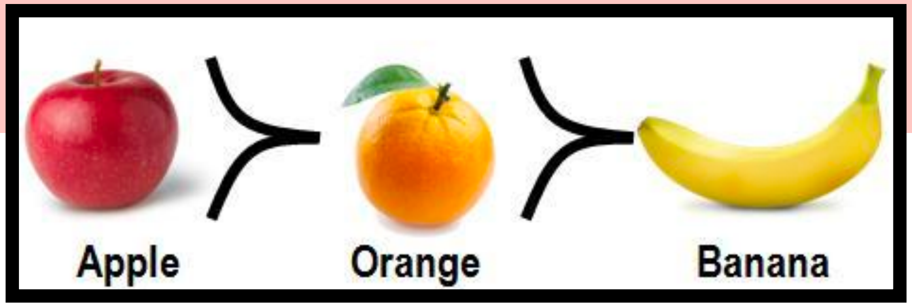


Recommendation

Which item should we offer?

Marketing

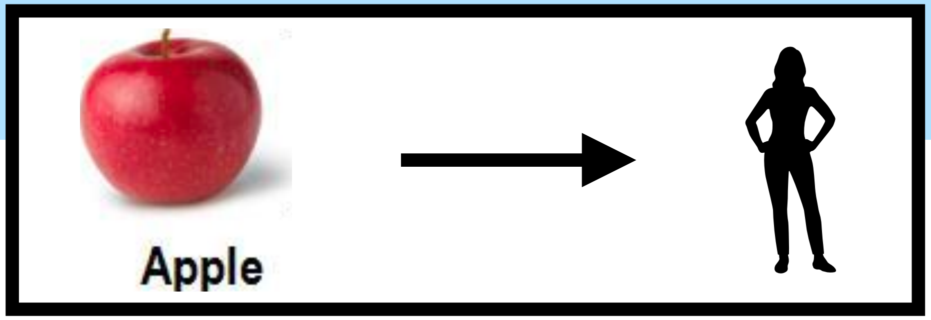
Which product should we advertise?

Medicine

Which procedure should we perform?

Public Policy

Which project should we fund?



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LAHSA

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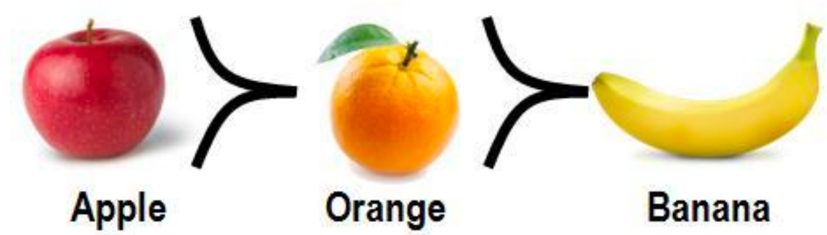

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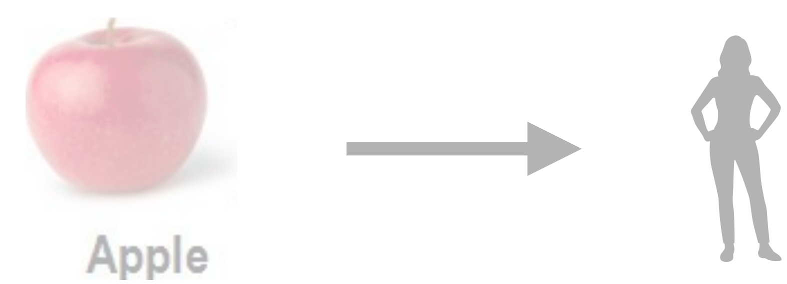

- Elicitation + Recommendation

Preferences

We have a set of **items** $\mathcal{X}$

- products
- policies
- procedures

$\mathcal{X}$ Could be **discrete** (we have a fixed set of items)

$$\mathcal{X} = \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^N\}$$

… or **continuous** (we can create items)

$$\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^J \mid \boldsymbol{x}^\top \boldsymbol{b} \leq \boldsymbol{c}\}$$

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For two items $(\boldsymbol{x}^A, \boldsymbol{x}^B)$…

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("$\succeq$" means "either $\succ$ or $\sim$")

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Aside

Cardinal preferences tell us *how much* one item is preferred over another.

Ordinal preferences only tell us *which* of two items is more-preferred.

Preferences

Representation Theorem

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For all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, either $\boldsymbol{x} \preceq \boldsymbol{y}$, $\boldsymbol{x} \succeq \boldsymbol{y}$, or both.

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If $\boldsymbol{x} \succeq \boldsymbol{y}$ and $\boldsymbol{y} \succeq \boldsymbol{z}$, then $\boldsymbol{x} \succeq \boldsymbol{z}$.

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C. and *continuous* <== (only for continuous $\mathcal{X}$)

If $\boldsymbol{x} \succeq \boldsymbol{y}$, then there is a neighborhood $B_{\boldsymbol{x}}$ $(B_{\boldsymbol{y}})$ around $\boldsymbol{x}$ $(\boldsymbol{y})$ such that $\boldsymbol{x}' \succeq \boldsymbol{y}'$ for all $\boldsymbol{x}' \in B_{\boldsymbol{x}}$ and $\boldsymbol{y}' \in B_{\boldsymbol{y}}$.

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Debreu's Representation Theorem:

For a topological space* $\mathcal{X}$, if preference relation $\succeq$ is **complete, transitive, and continuous,** then it is representable by a utility function $u : \mathcal{X} \to \mathbb{R}$, s.t.:

$$\boldsymbol{x}^A \succeq \boldsymbol{x}^B \iff u(\boldsymbol{x}^A) \geq u(\boldsymbol{x}^B)$$

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Preferences

Representation Theorem: Example

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Ordinal Preferences:

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Usually we can't learn an agent's preference relation, instead we model it.

Preferences

Modeling Preferences

Ordinal Preferences $\succeq$

$\Updownarrow$

Utility Function $u(\cdot)$

Preferences

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Usually not practical to learn a complete preference relation $\succeq$.

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Instead, we *approximate* agent preferences…
Two approaches, in general:

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Two approaches, in general:

Probabilistic

Learn a *distribution over* $\succeq$ or $u(\cdot)$

- Bradley-Terry
- Plackett-Luce
- Thurstonian models
- …

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Set-Based (today's focus)

Learn a *set* that contains $\succeq$ or $u(\cdot)$

Preferences

Linear Utility Functions

Assumptions:

- Agent has a utility function $u(\boldsymbol{x})$…
- Which is linear in each item's features

$$u(\boldsymbol{x}) = \boldsymbol{u}^\top \boldsymbol{x}$$

Item features ($\boldsymbol{x}$)

Uncertain vector of utility function coefficients ($\boldsymbol{u}$)

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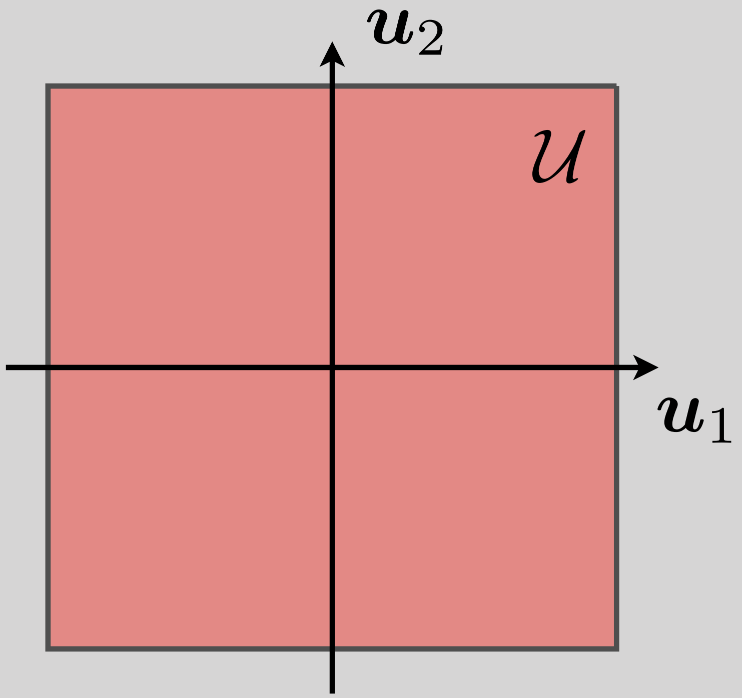
Uncertain vector of utility function coefficients

Set-based uncertainty:

- Agent utility vector $\boldsymbol{u}$ belongs to an *uncertainty set* $\mathcal{U}$
- Our goal is to narrow down $\mathcal{U}$



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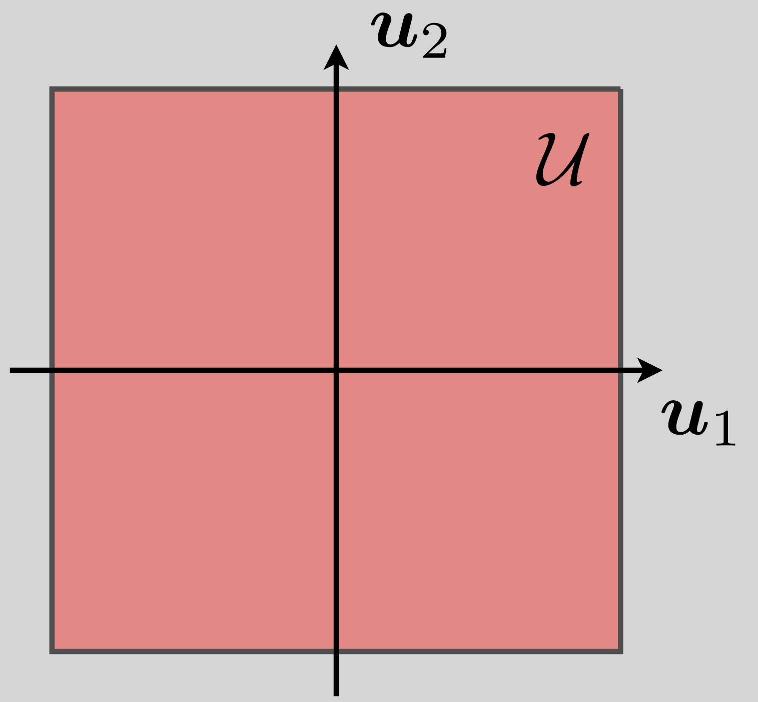

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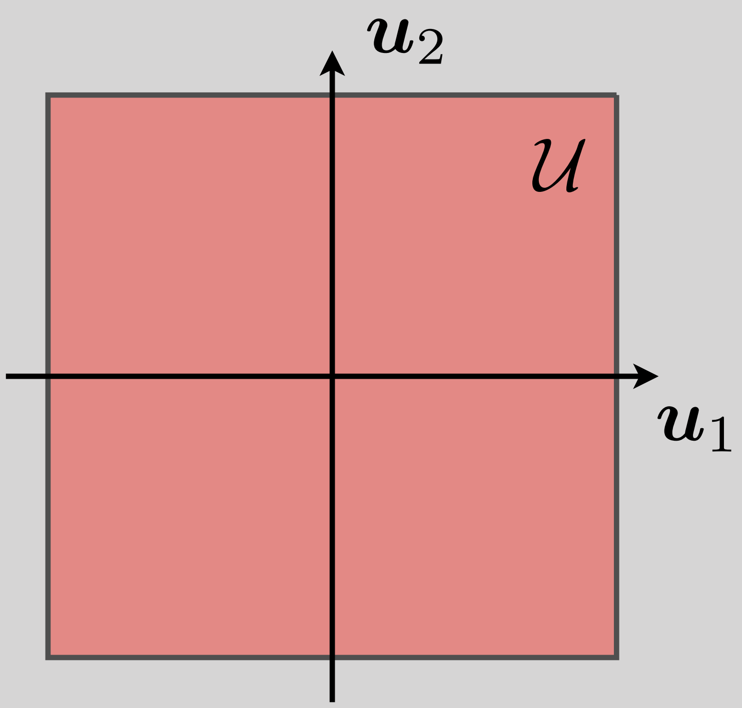


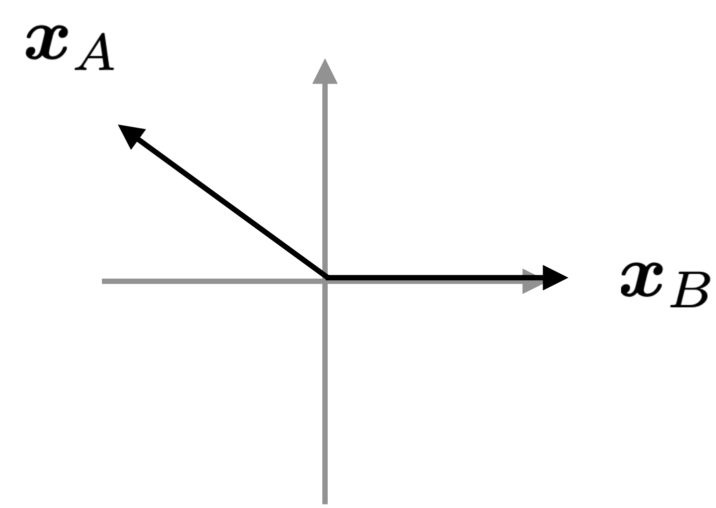
- $u(\mathbf{x})$ cannot be observed directly (***in our setting***)
- Instead, ask **pairwise comparisons** "do you prefer item $\boldsymbol{x}_A$ or $\boldsymbol{x}_B$?")

Preference Elicitation

Example: One Pairwise Comparison

Question 1: Do you prefer…

$$\boldsymbol{x}_A = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \textbf{or} \quad \boldsymbol{x}_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



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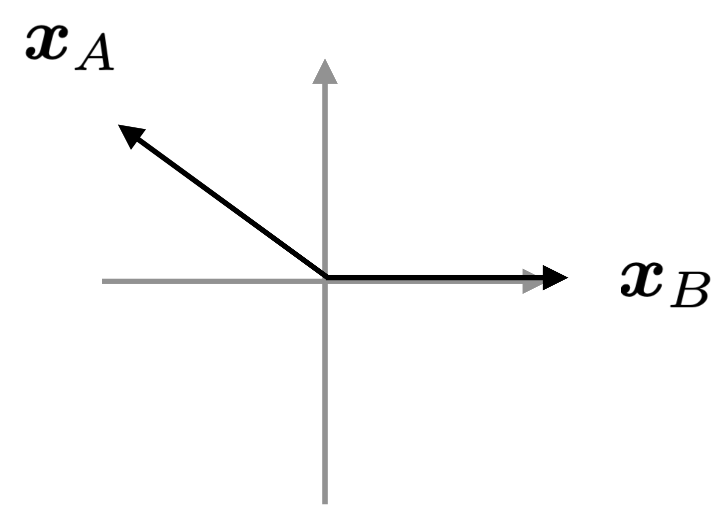

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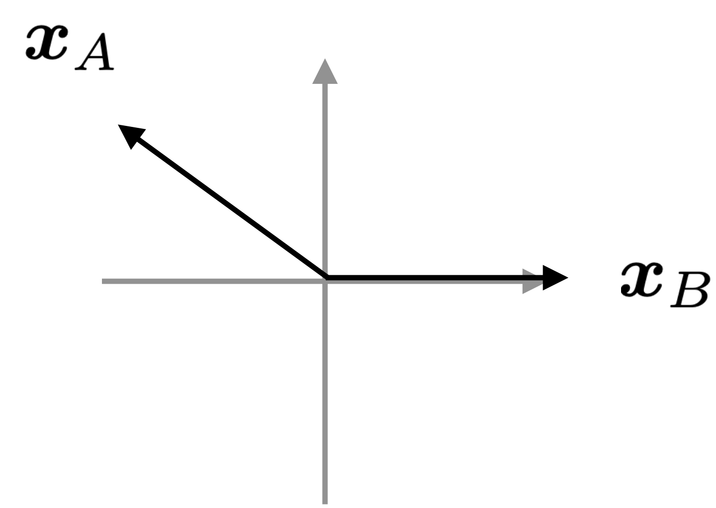

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Answer 1: $\boldsymbol{x}_A$

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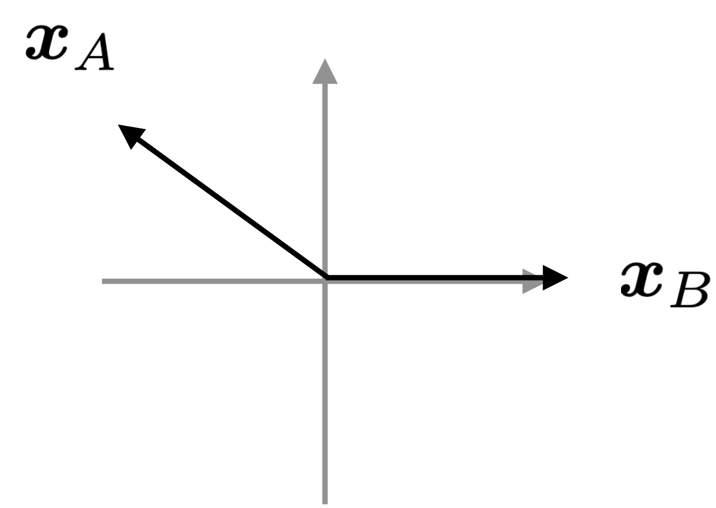

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Answer 1: $\boldsymbol{x}_A$ ➡ $\boldsymbol{u}^\top (\boldsymbol{x}_A - \boldsymbol{x}_B) > 0$

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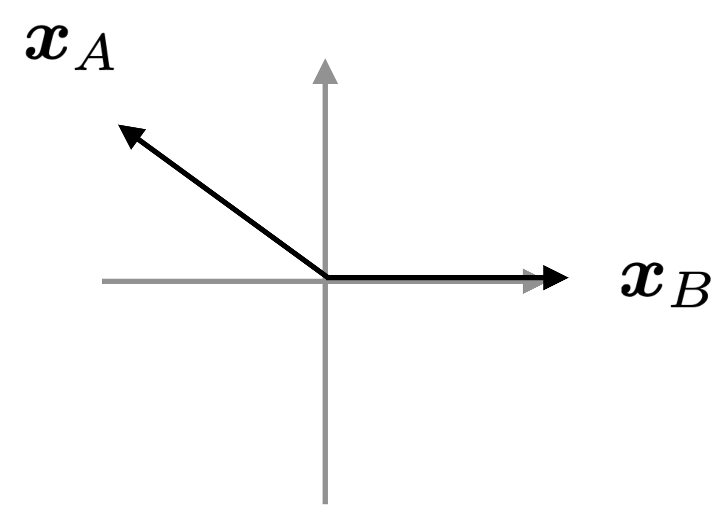

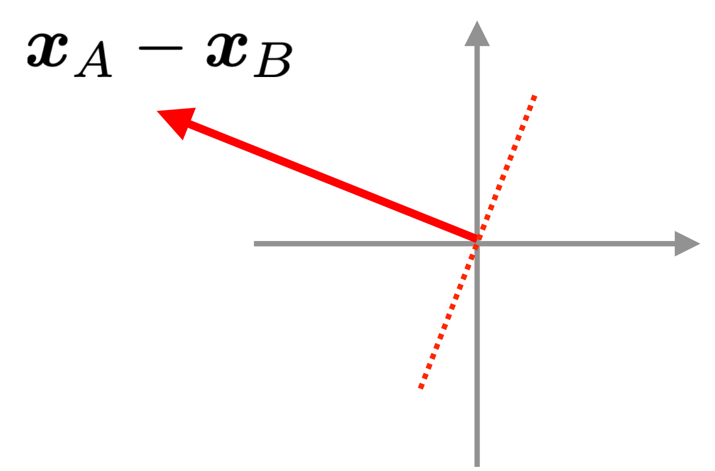

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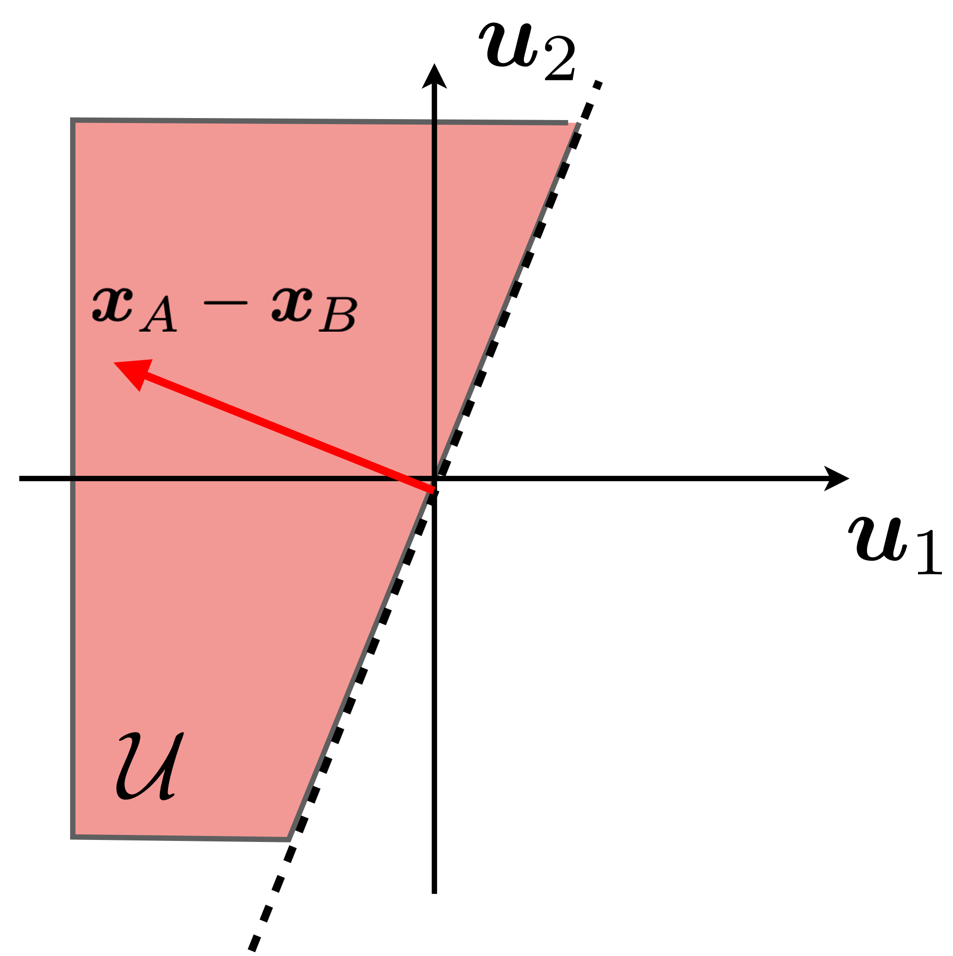
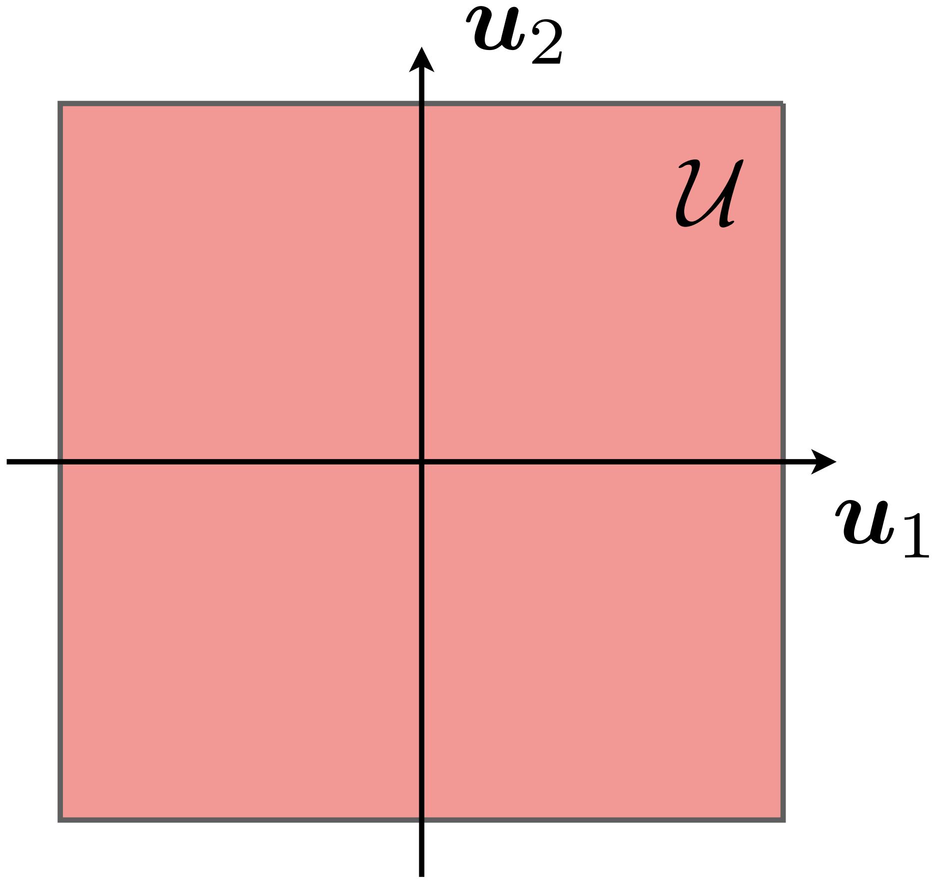


Preference Elicitation

Example: Many Pairwise Comparisons



Preference Elicitation

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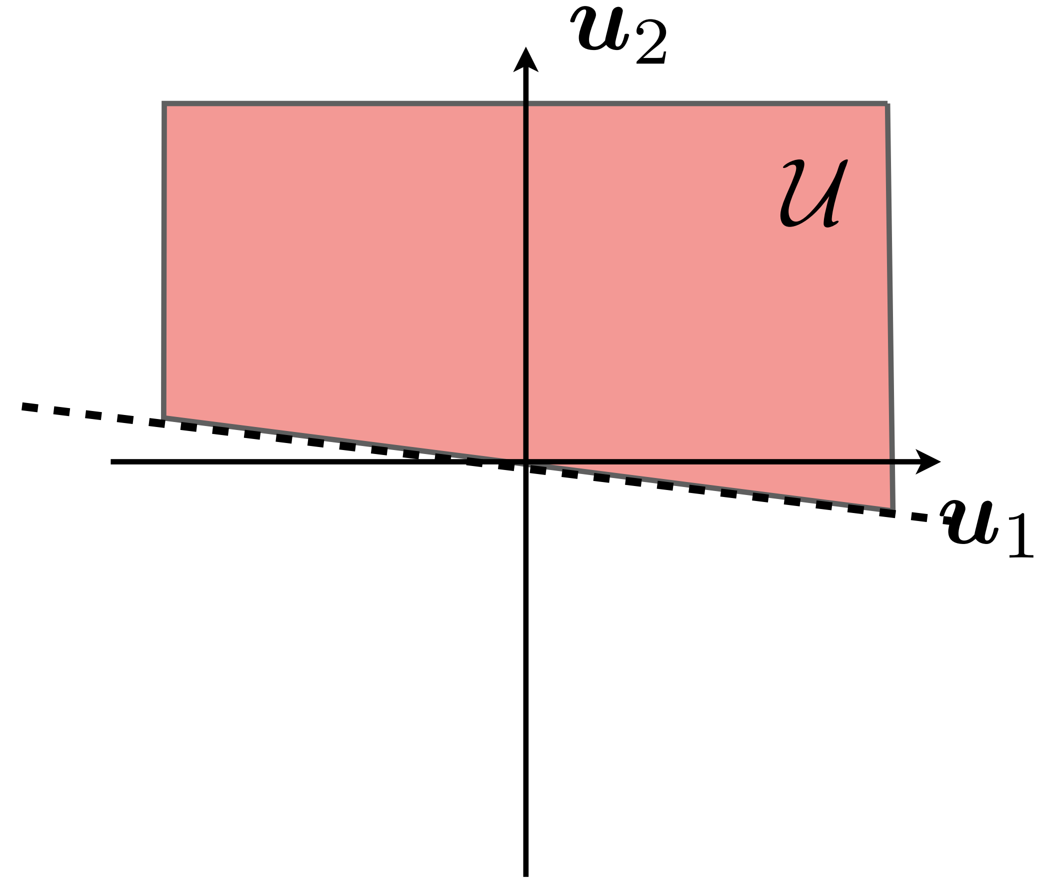
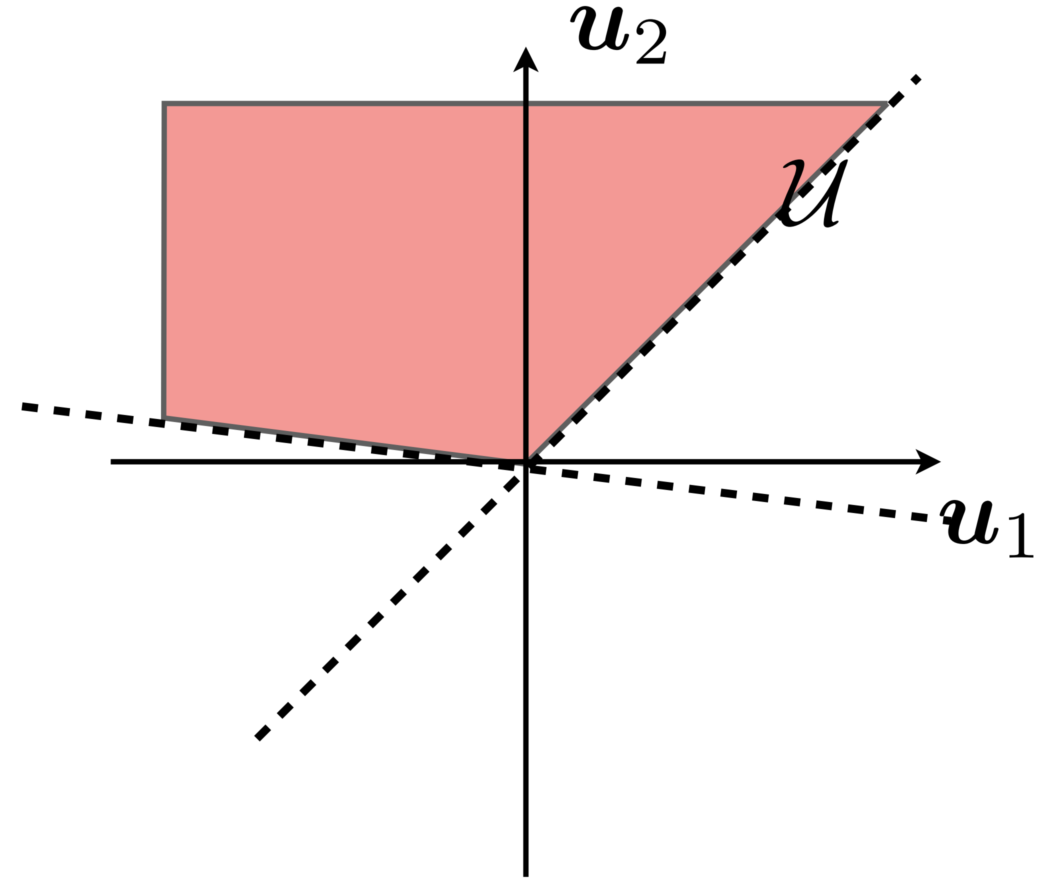


Preference Elicitation

Example: Many Pairwise Comparisons

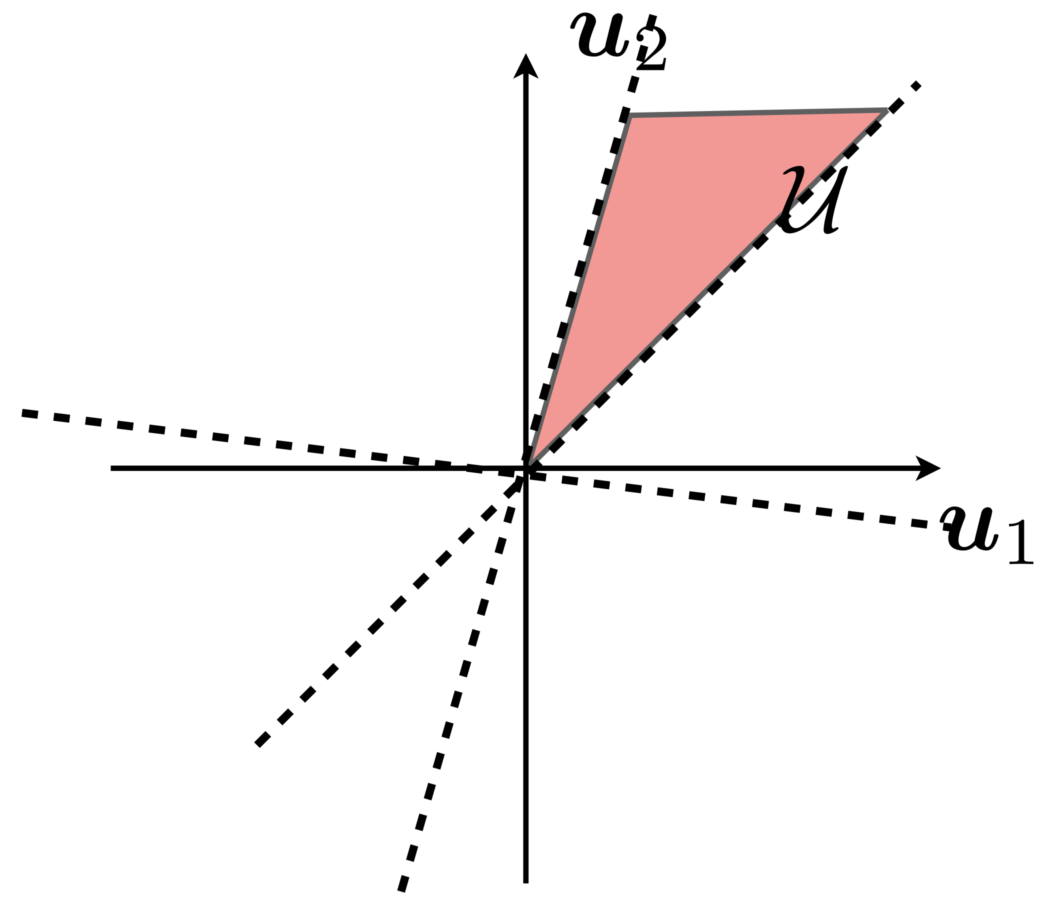
Question 1…
Question 2…



Preference Elicitation

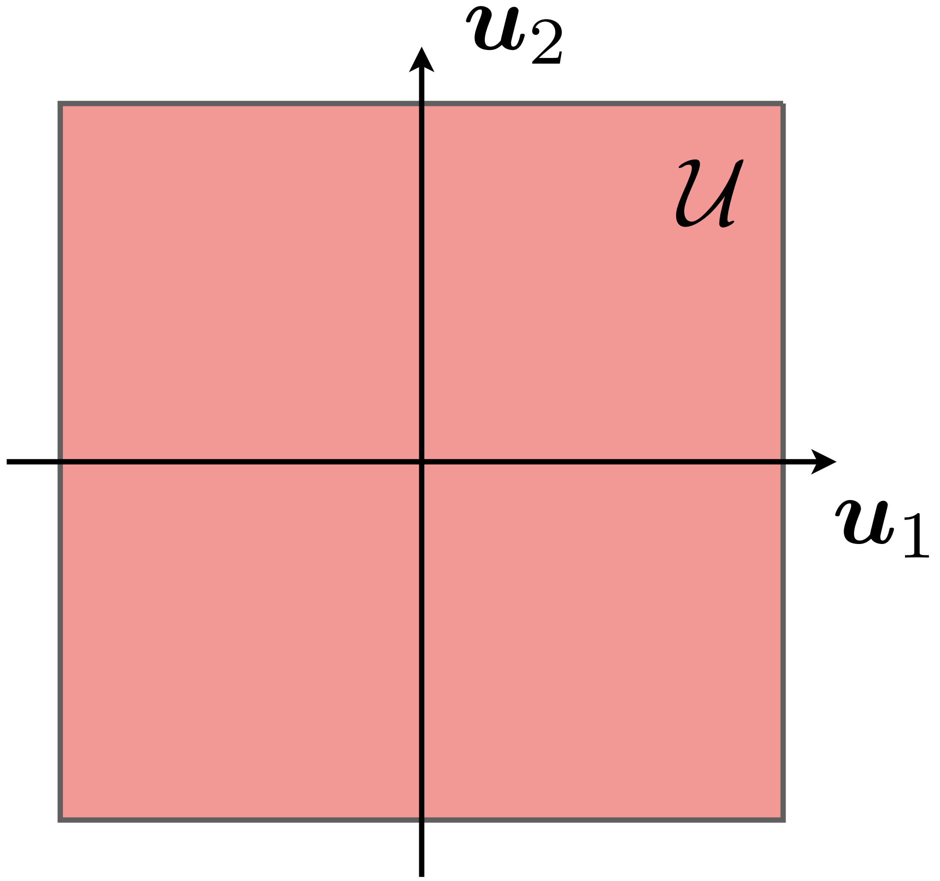
Example: Many Pairwise Comparisons

Question 1…
Question 2…
Question 3…



Preference Elicitation

Bad Example: Many Pairwise Comparisons



Preference Elicitation

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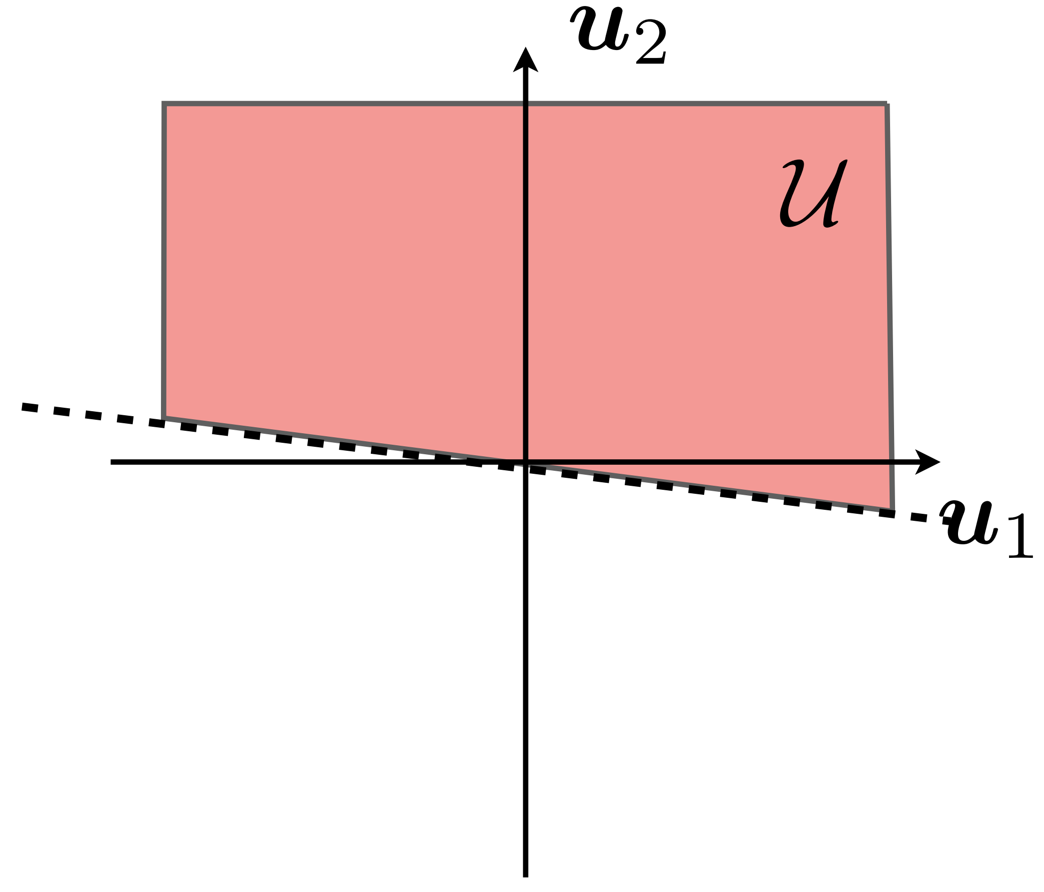

Preference Elicitation

Bad Example: Many Pairwise Comparisons

Question 1…
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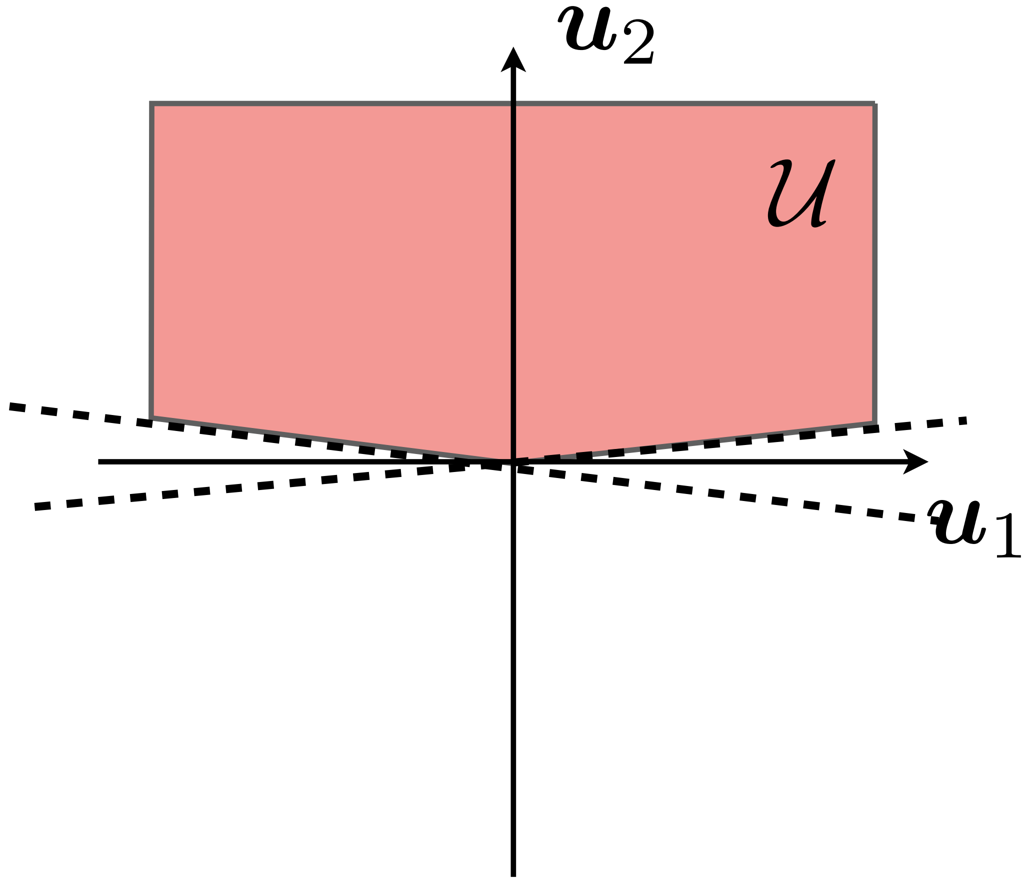
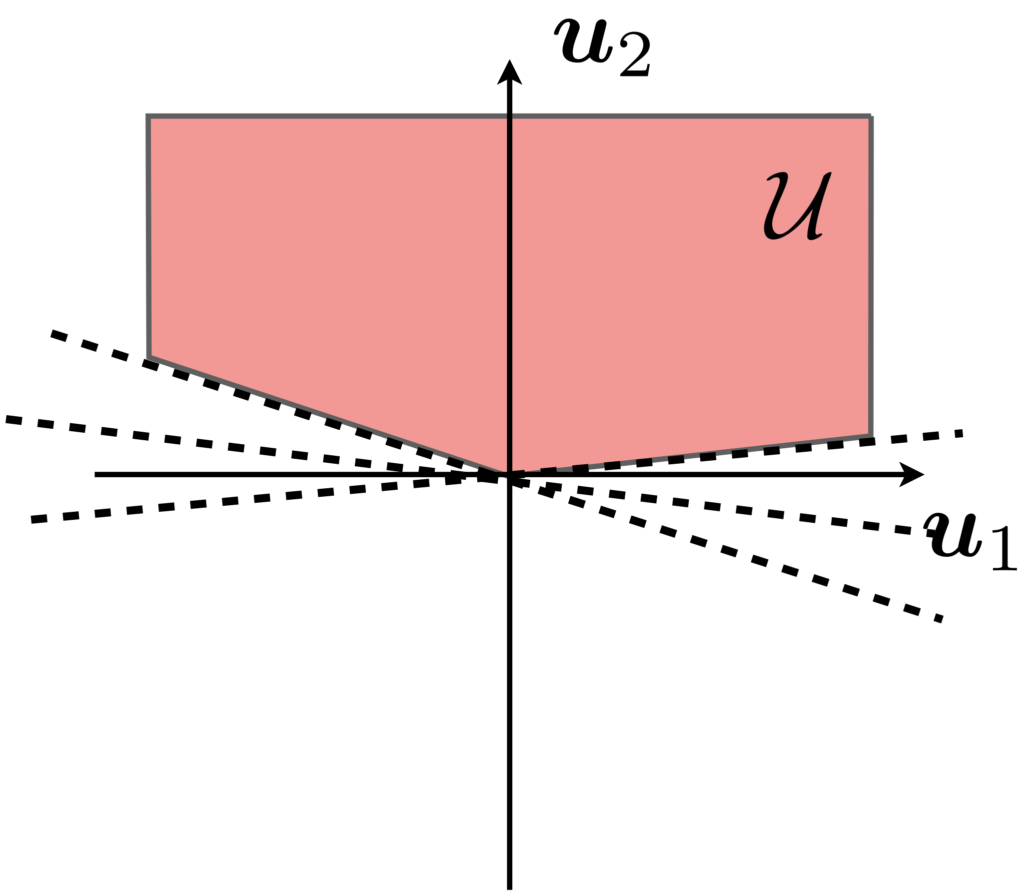


Preference Elicitation

Bad Example: Many Pairwise Comparisons

Problem:

Which comparisons should we choose to elicit responses from the agent?



Preference Elicitation

Heuristics

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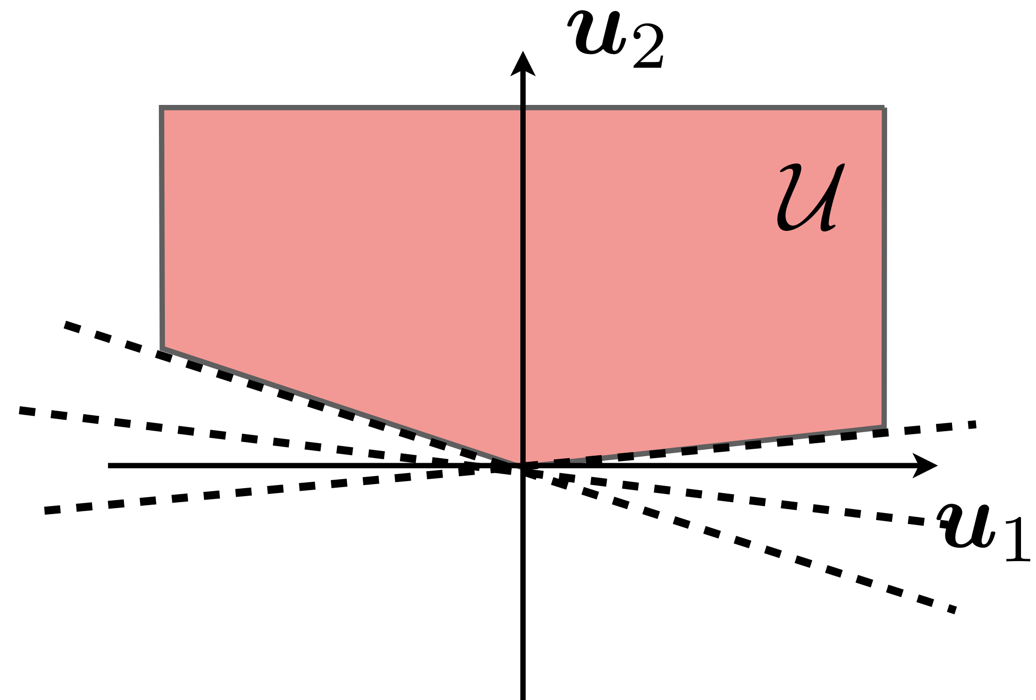

Heuristic Methods

Select queries to minimize the size of $\mathcal{U}$

- Toubia et al. [Fast Polyhedral Adaptive Conjoint Estimation. *Marketing Science*]
- Toubia et al. [Polyhedral Methods for Adaptive Choice-Based Conjoint Analysis. *Journal of Marketing Research*]

Geometric heuristics

- Bertsimas et al. [Constraint-based optimization and utility elicitation using the minimax decision criterion.]
- Bertsimas et al. [Learning Preferences Under Noise and Loss Aversion: An Optimization Approach. *Operations Research*]

Preference Elicitation

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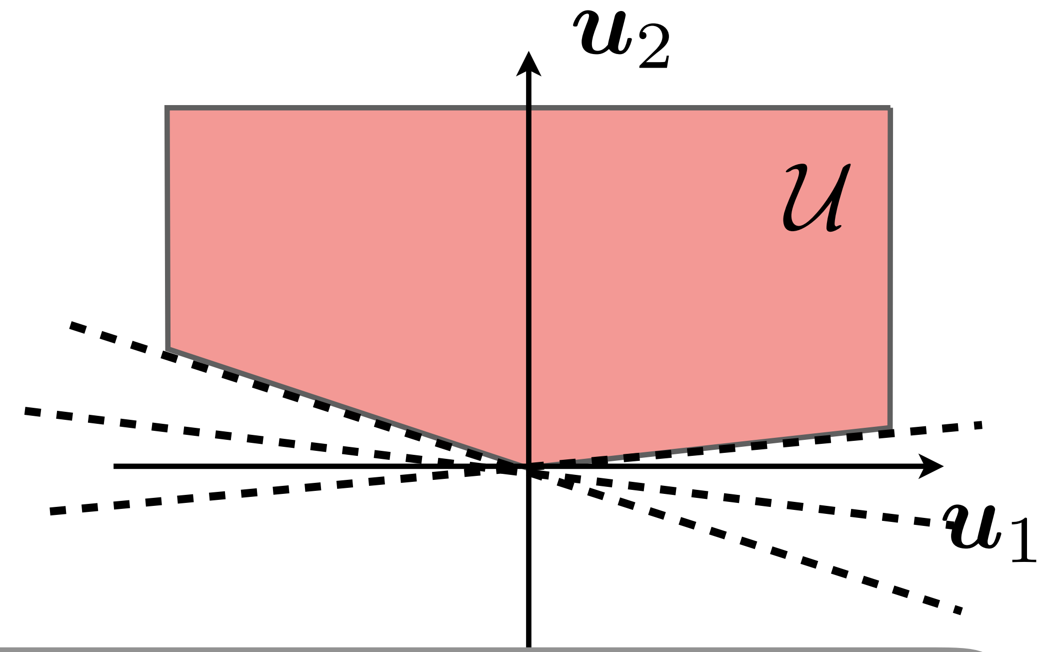
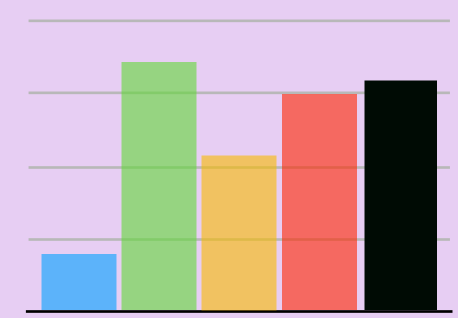


Heuristics are nice, but they don't get at our "final" goal: **recommendation**



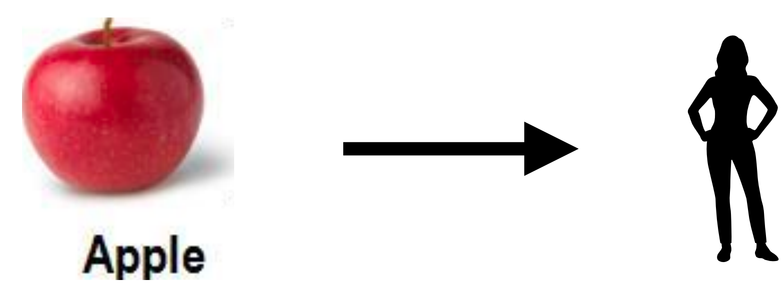
Which policy is **best**?

Outline

- Application: Learning an Objective Function

- Preference Elicitation

- Recommendation Under Uncertainty



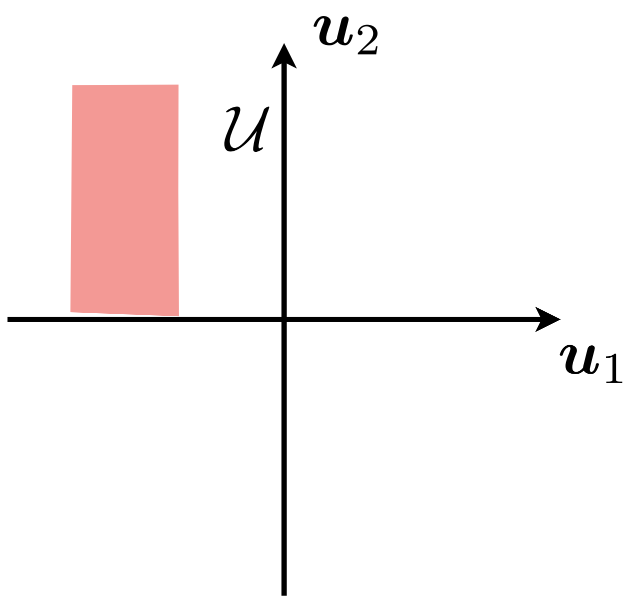
- Elicitation + Recommendation

Recommendation

Decision Criteria

We are **uncertain** about the agent's preferences…

How do we recommend a "good" item?



Recommendation

Decision Criteria

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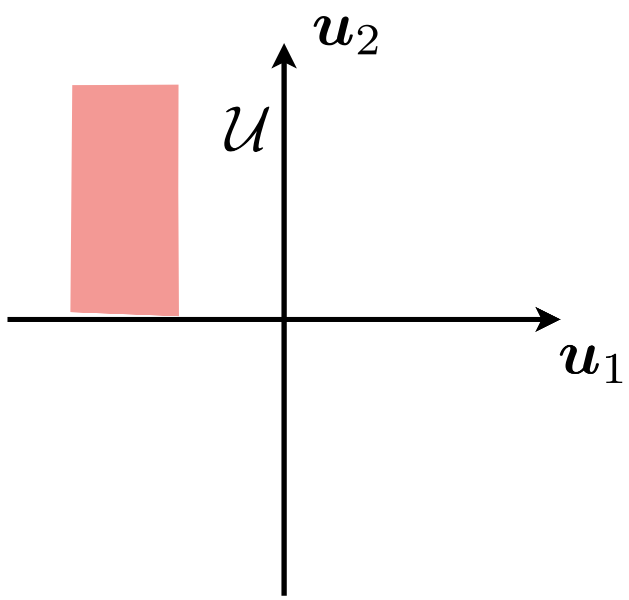

Maximax (optimistic)

*Select the item with the greatest utility in the **best case***

Recommendation

Decision Criteria

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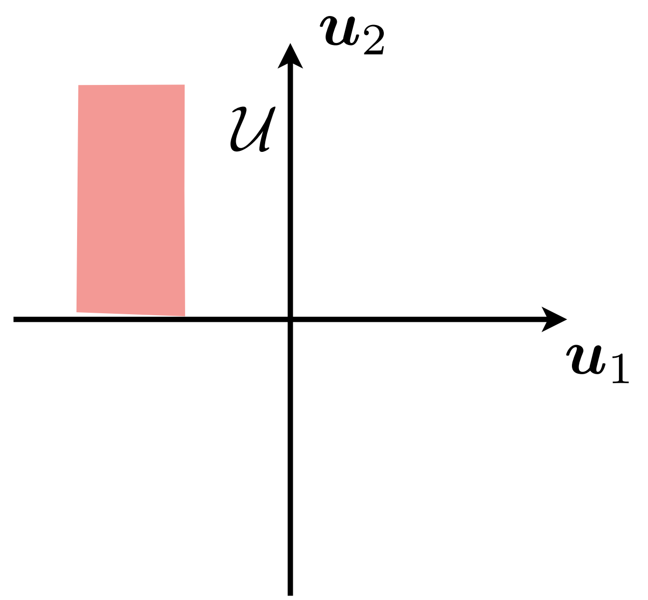

Maximax (optimistic)

Select the item with the greatest utility in the ***best case***

Maximin (pessimistic)

Select the item with the largest utility in the ***worst case***

Recommendation

Decision Criteria

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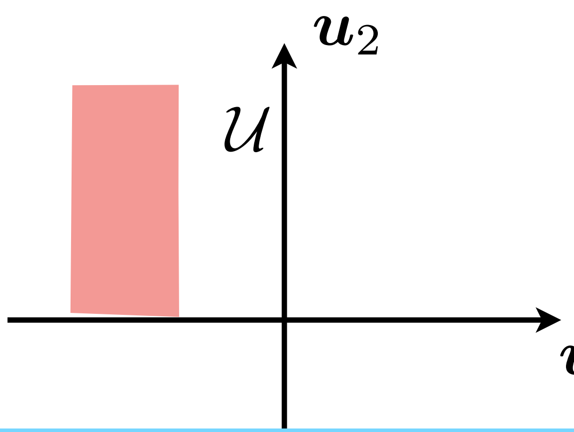

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Minimax Regret

Select the item which minimizes ***worst-case regret***

Recommendation

Decision Criteria

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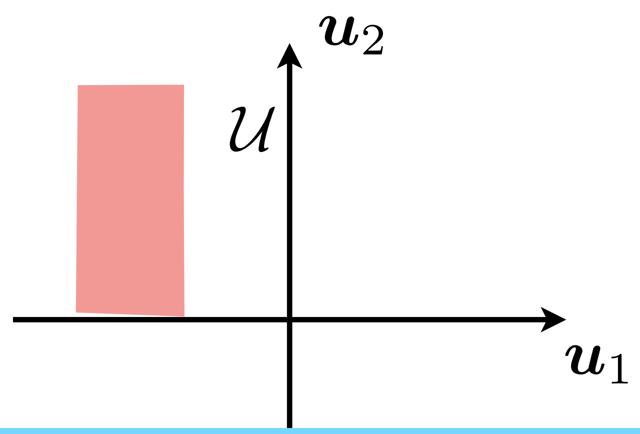

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Maximin (pessimistic)

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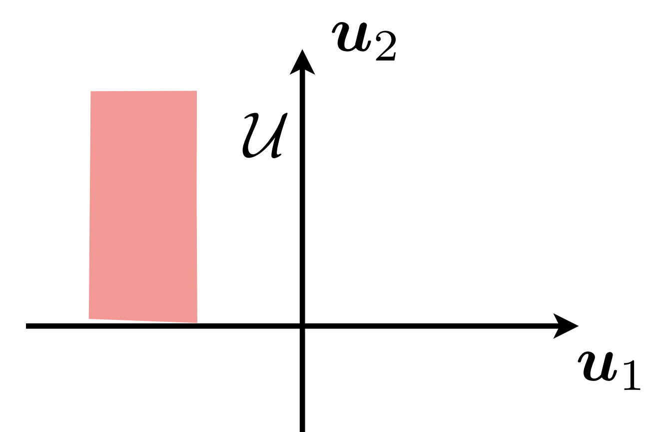

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Maximize…

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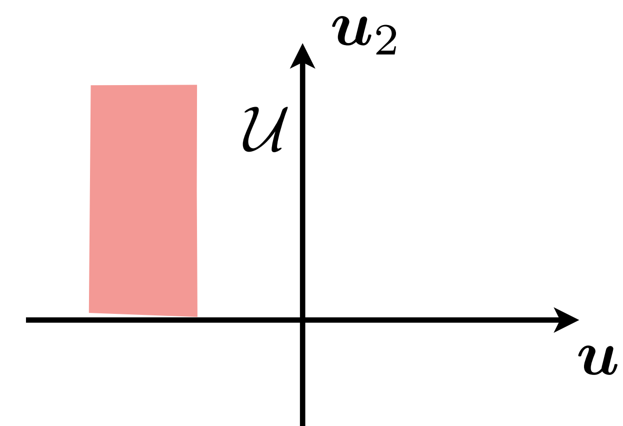

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Maximize… the best-case *u(x)*

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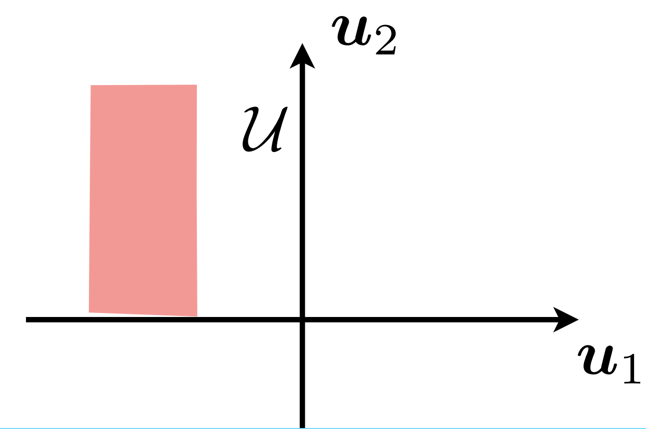

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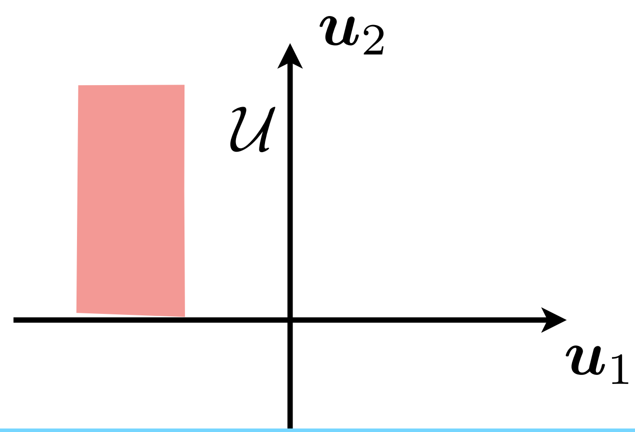

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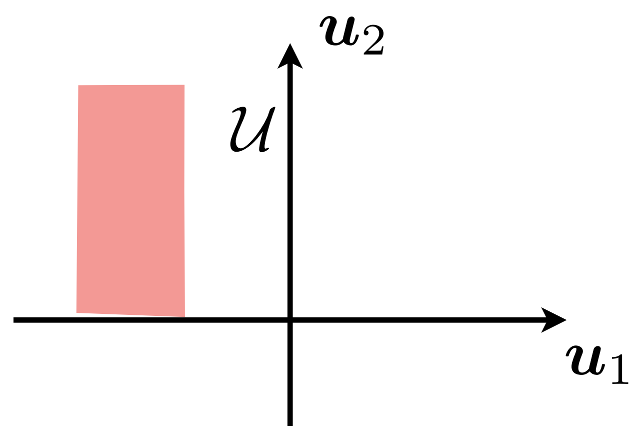

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Minimax Regret

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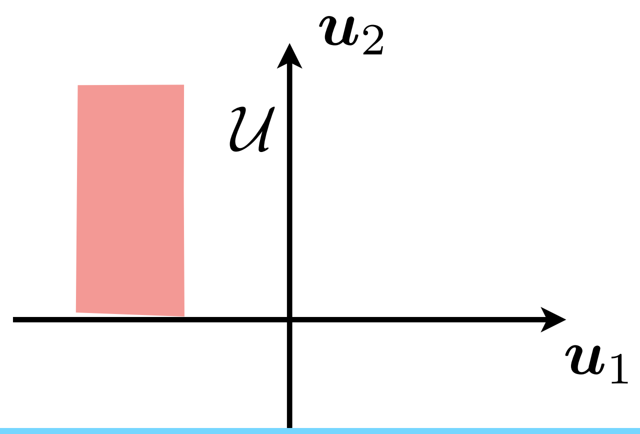

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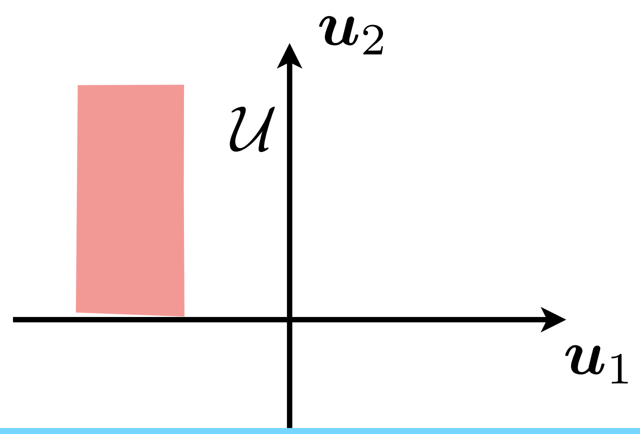


Minimax Regret

Select the item which minimizes ***worst-case regret***

Regret of recommending item ***x***
to agent with u-vec ***u***:

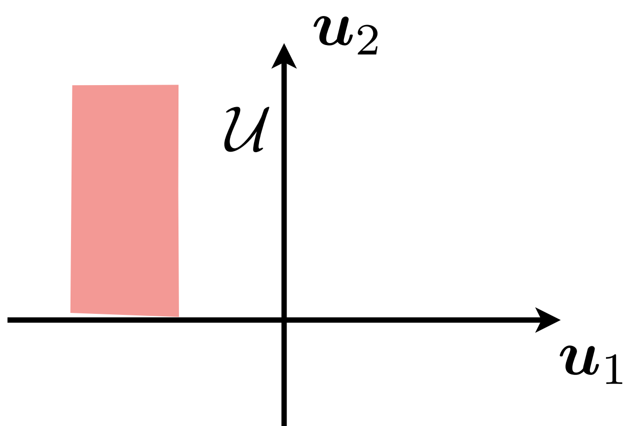
$$R(\boldsymbol{u}, \boldsymbol{x}) \equiv \left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x}$$

Recommendation

Minimax Regret

We are **uncertain** about the agent's preferences…

How do we recommend a "good" item?



Minimax Regret

Select the item which minimizes ***worst-case regret***

Regret of recommending item ***x*** to agent with u-vec ***u***:

$$R(\boldsymbol{u}, \boldsymbol{x}) \equiv \left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x}$$

Recommend the item that *minimizes* ***worst-case*** **regret:**

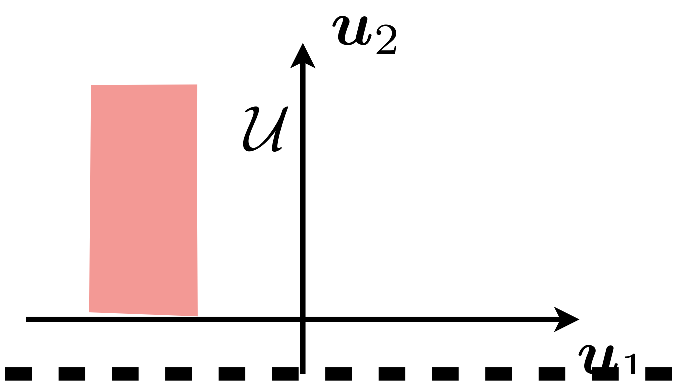
$$\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x} \in \mathcal{X}} \max_{\boldsymbol{u} \in \mathcal{U}} R(\boldsymbol{x}, \boldsymbol{u})$$

Recommendation

Decision Criteria

We are **uncertain** about the agent's preferences…

How do we recommend a "good" item?



"Robust" Approaches to Uncertainty

i.e., make the best recommendation in the *worst case*.

Maximin (pessimistic)

*Select the item with the largest utility in the **worst case***

$$\boldsymbol{x}^* \in \arg\max_{\boldsymbol{x}\in\mathcal{X}} \min_{\boldsymbol{u}\in\mathcal{U}} \boldsymbol{u}^\top \boldsymbol{x}$$

Minimax Regret

*Select the item which minimizes **worst-case regret***

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Recommendation

Decision Criteria: Example

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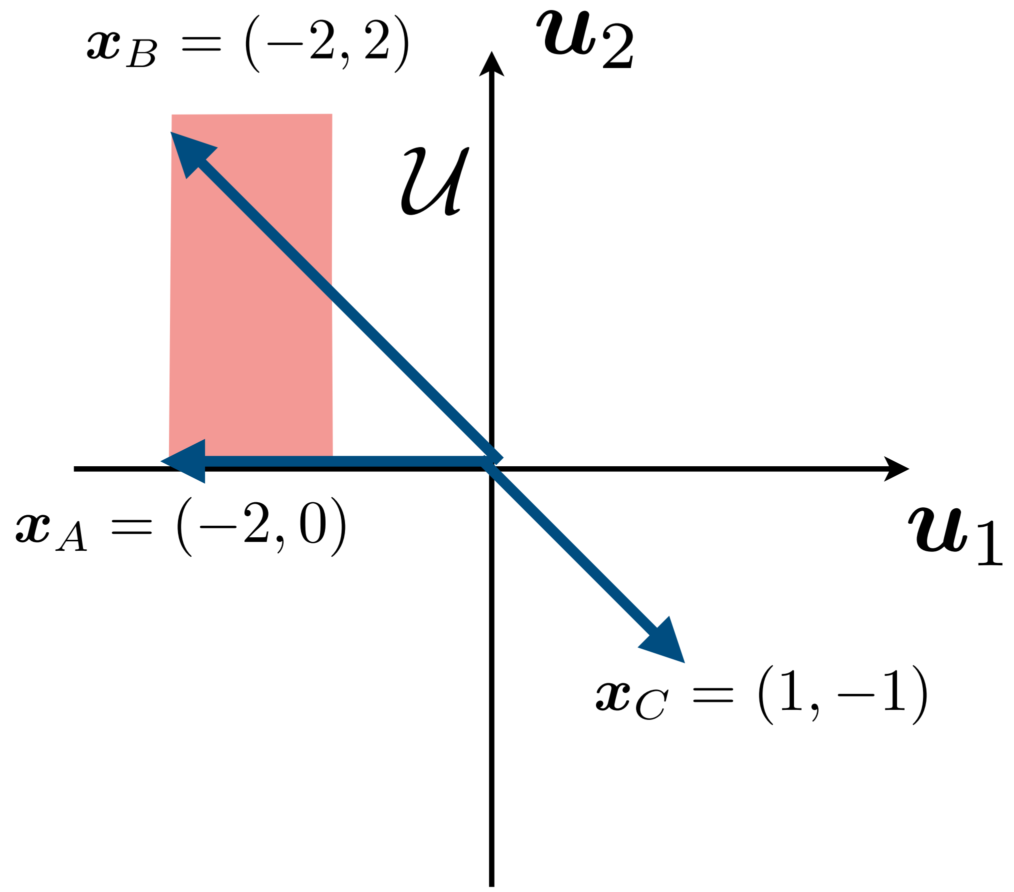

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

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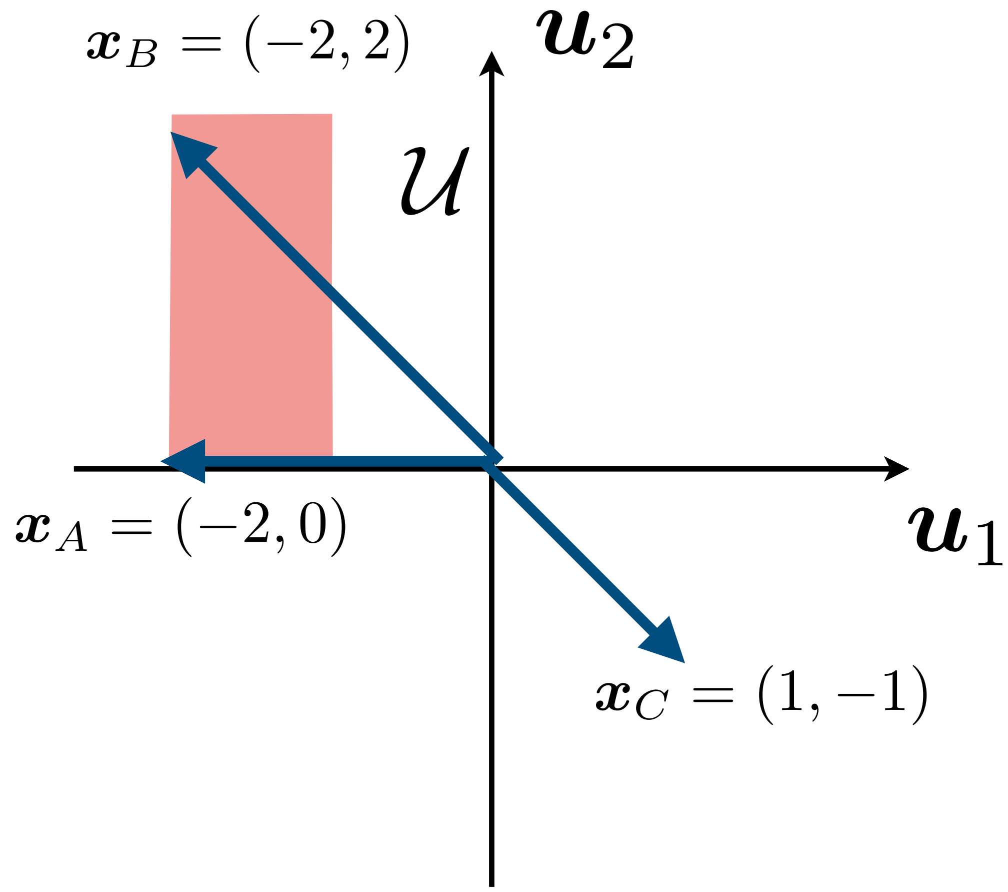

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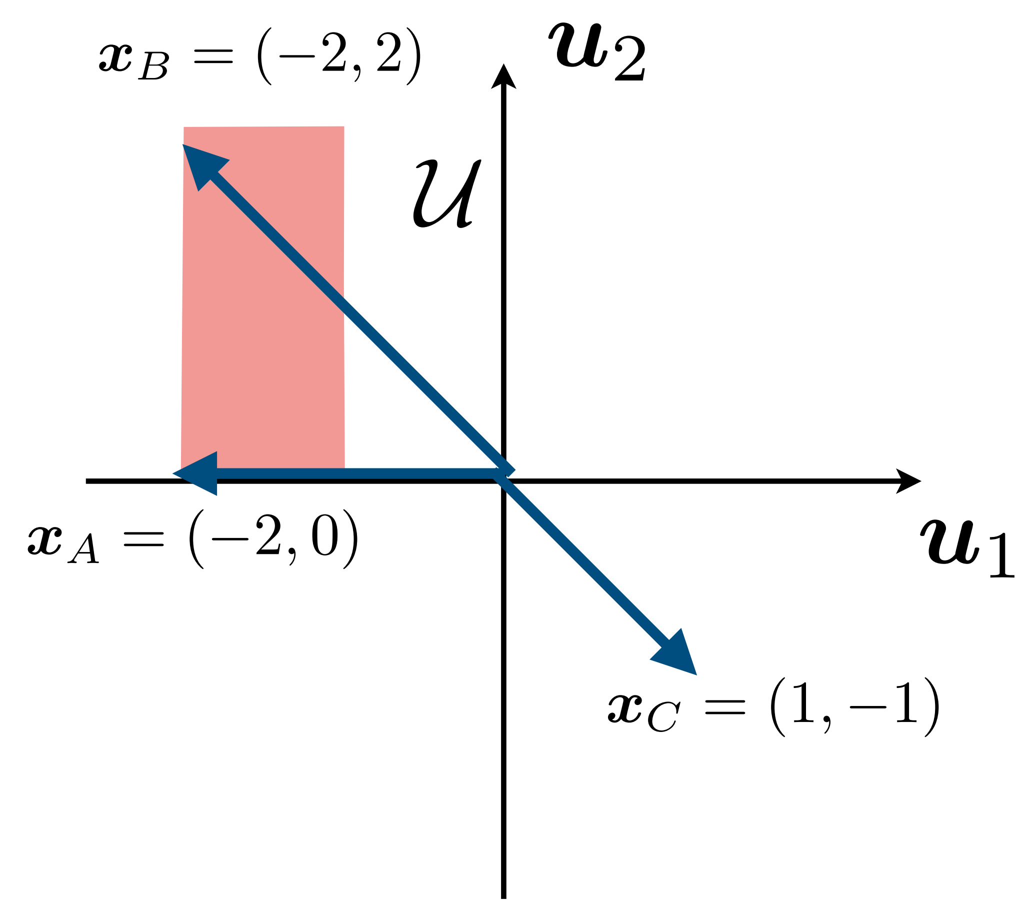
Best-case utilities:



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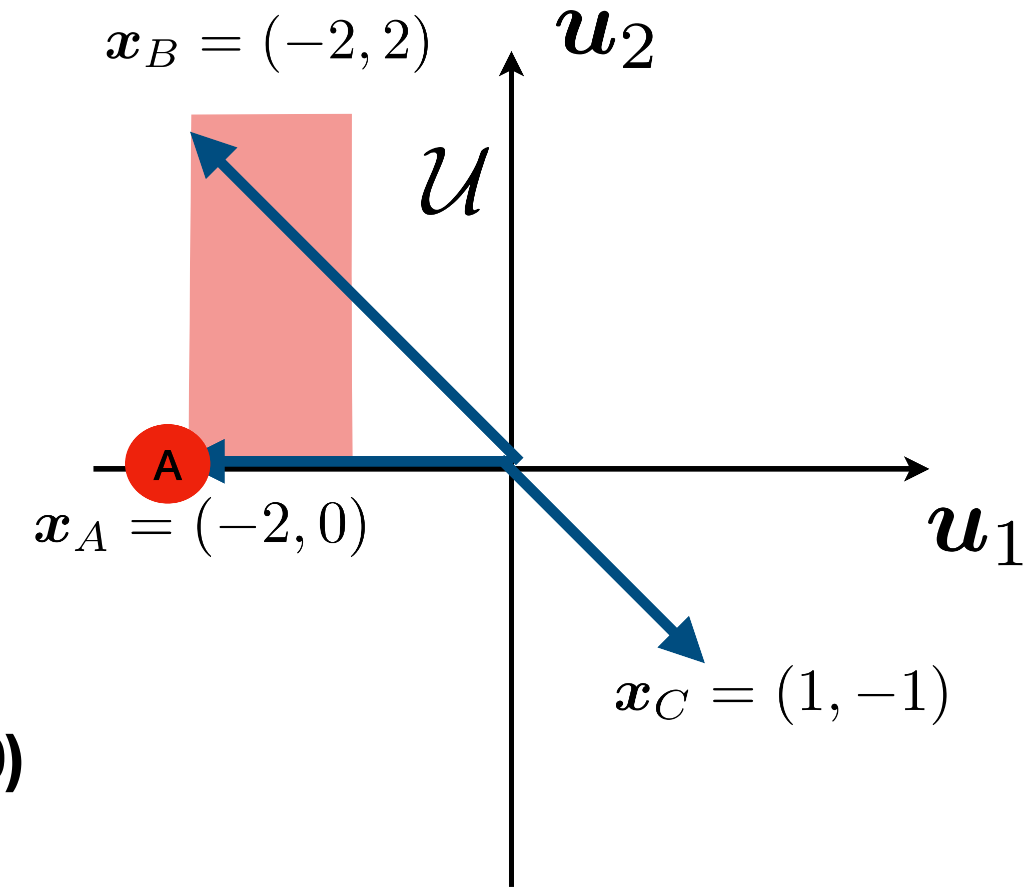

Best-case utilities:

$\boldsymbol{x}_A$ **: 4 , u = (-2, 0)**

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

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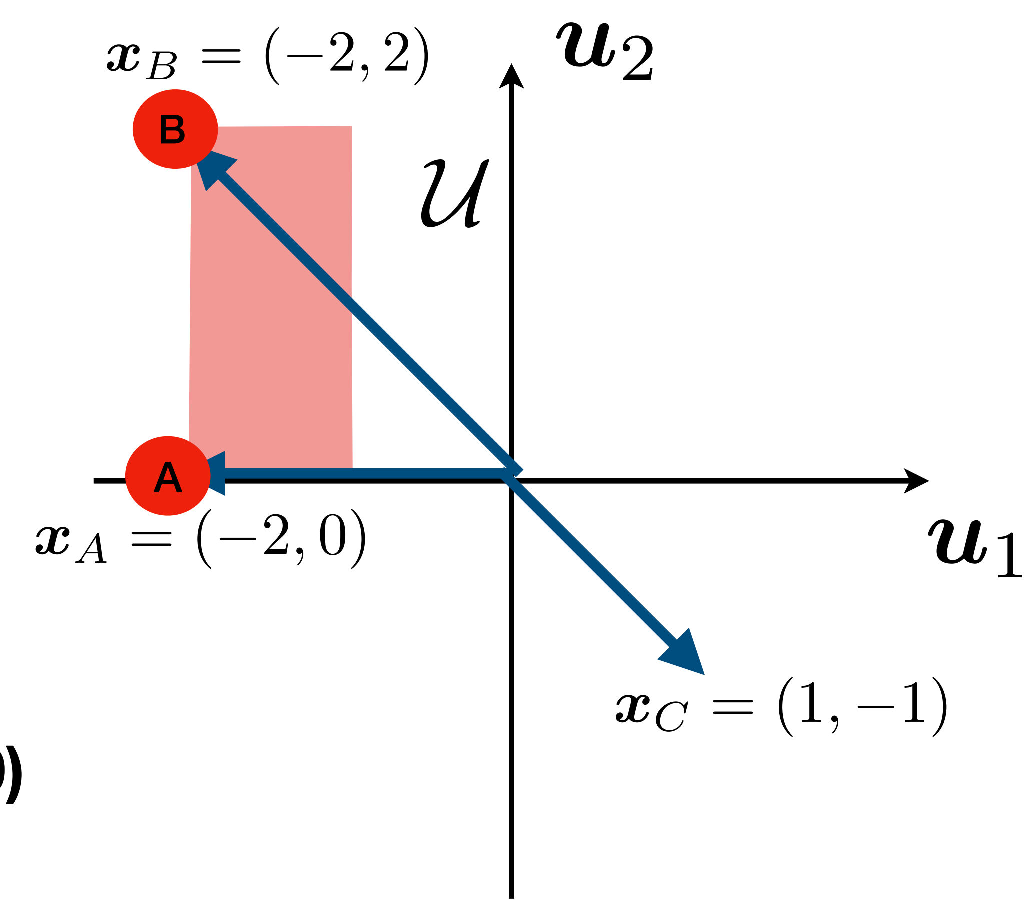

Best-case utilities:

$\boldsymbol{x}_A$: **4 , u = (-2, 0)**

$\boldsymbol{x}_B$: **8 , u = (-2, 2)**

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

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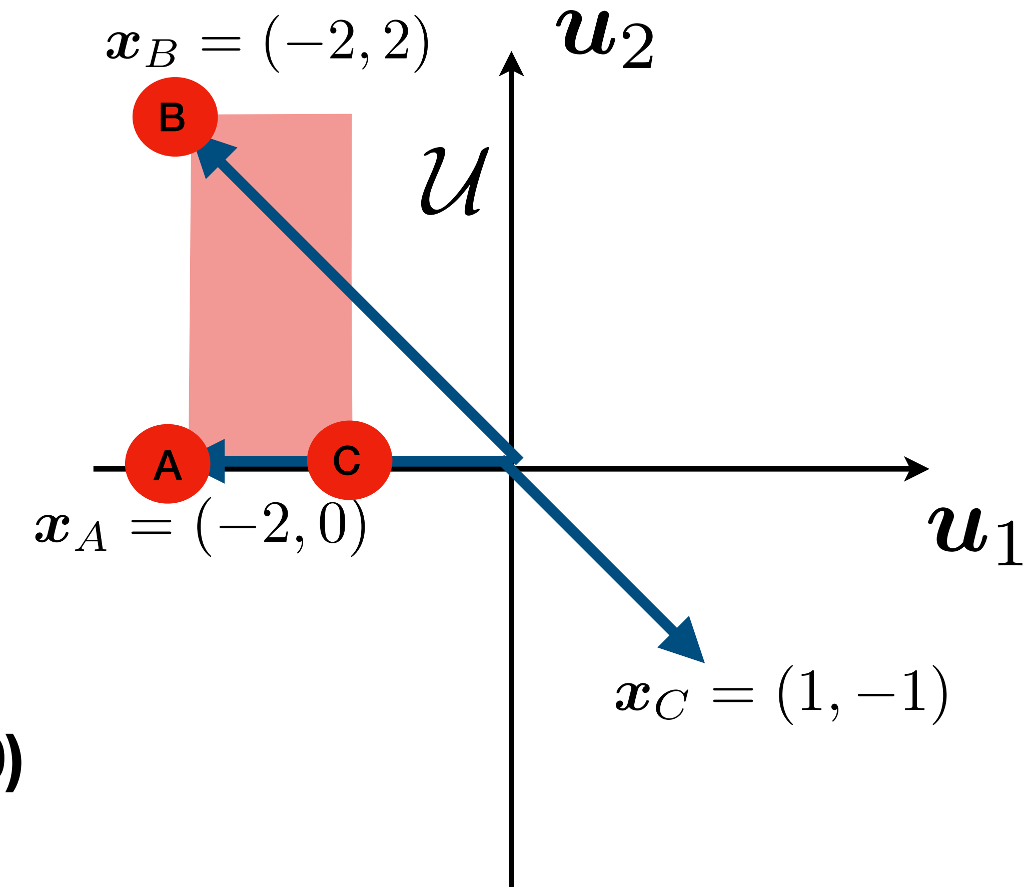

Best-case utilities:

$\boldsymbol{x}_A$: **4 , u = (-2, 0)**

$\boldsymbol{x}_B$: **8 , u = (-2, 2)**

$\boldsymbol{x}_C$: **-1 , u = (-1, 0)**

Recommendation

Decision Criteria: Example

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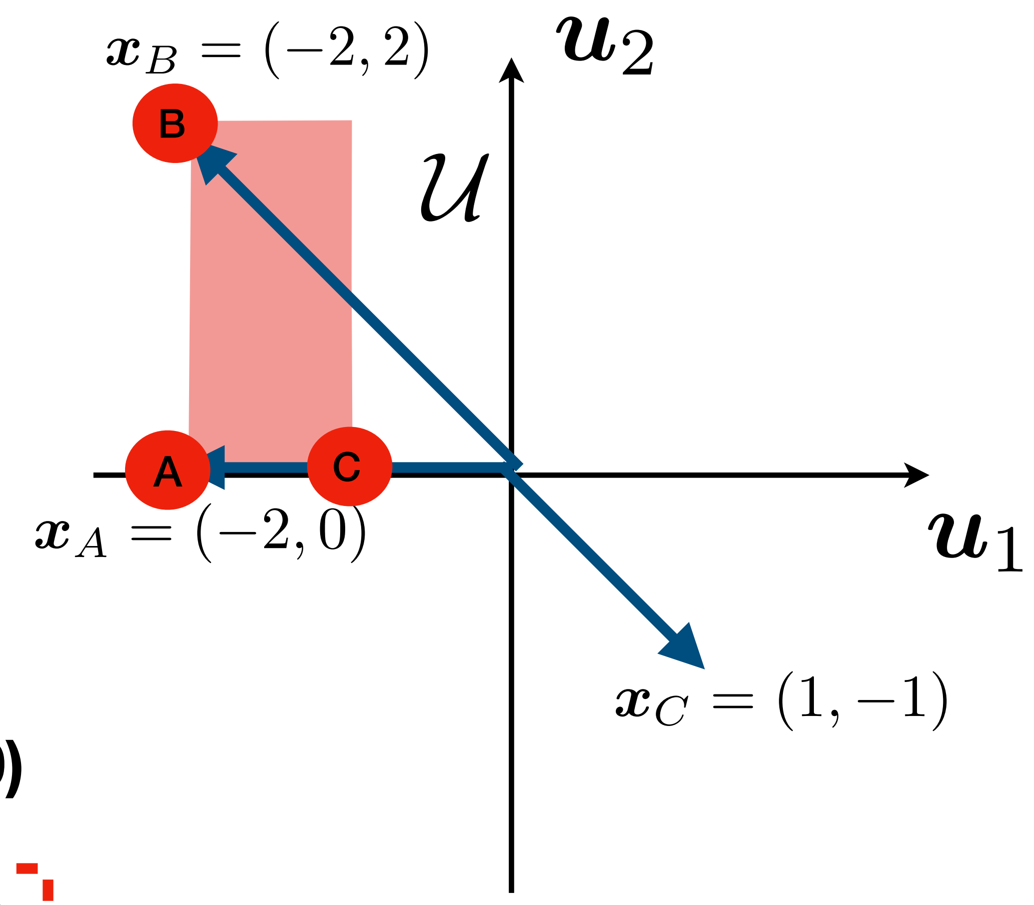

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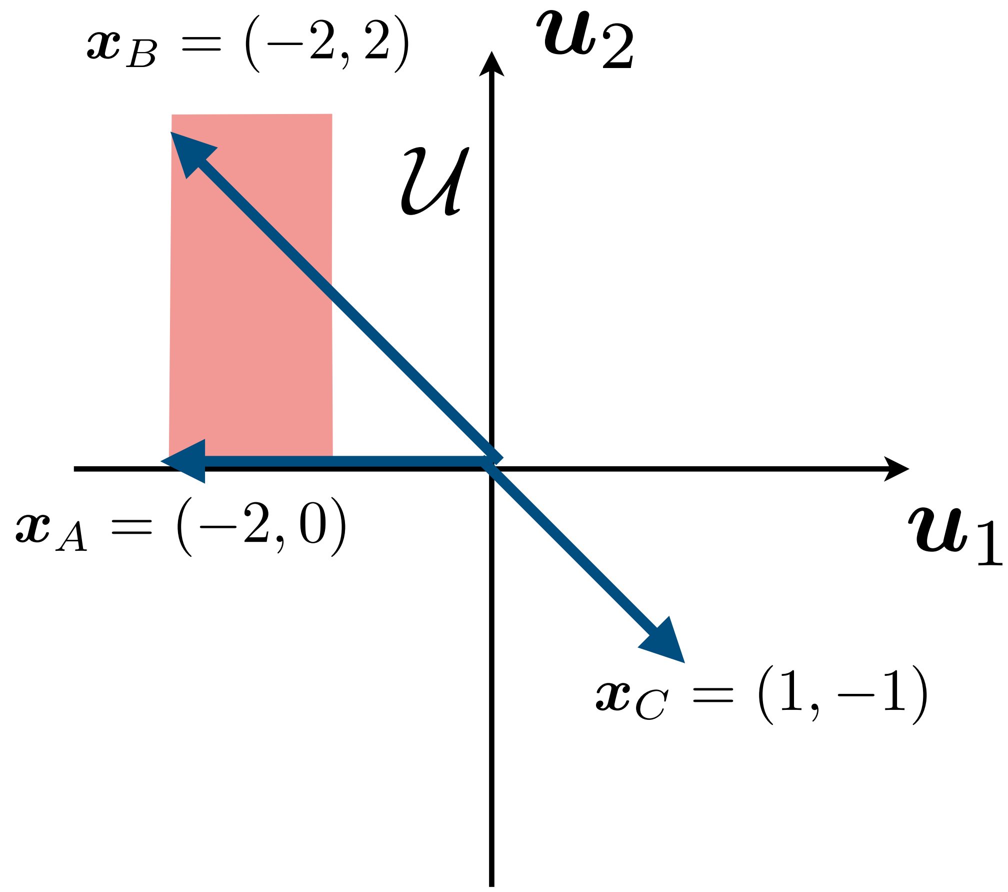

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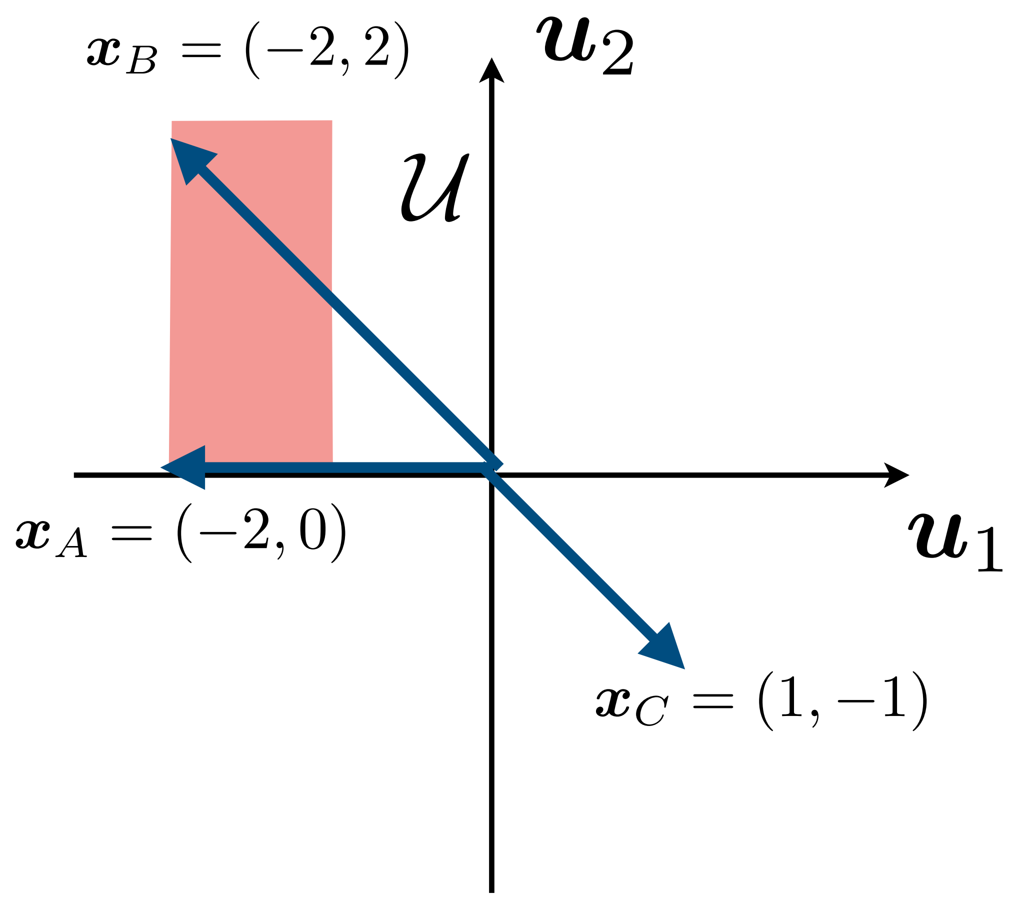

Worst-case utilities:

Recommendation

Decision Criteria: Example

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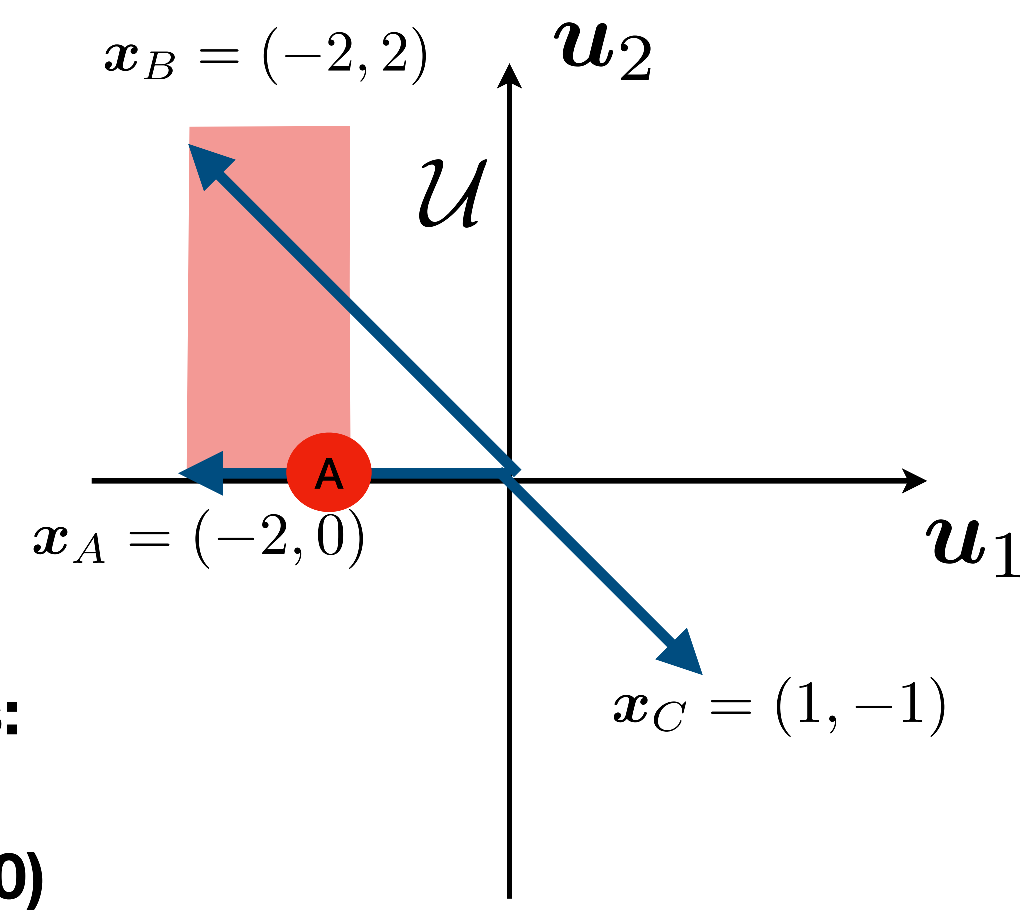

Worst-case utilities:

$\boldsymbol{x}_A$ **: 2 , u = (-1, 0)**

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

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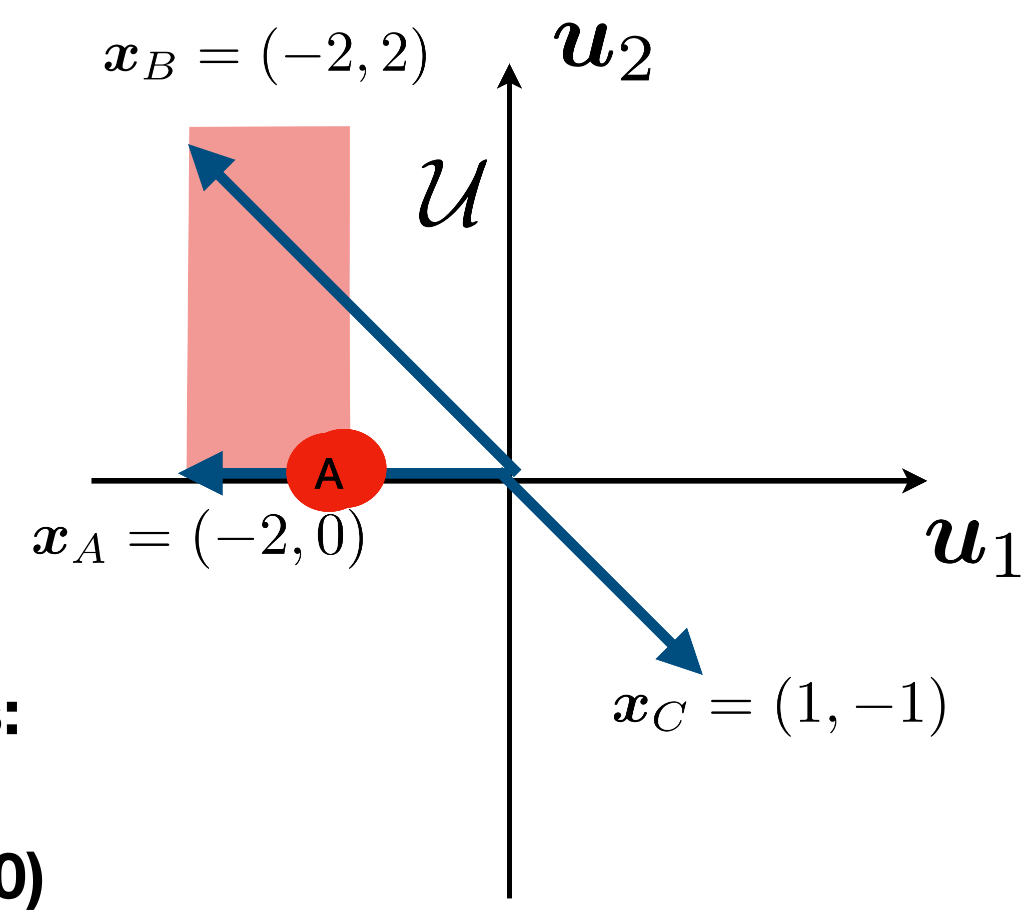

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$\boldsymbol{x}_B$ **: 2 , u = (-1, 0)**

Recommendation

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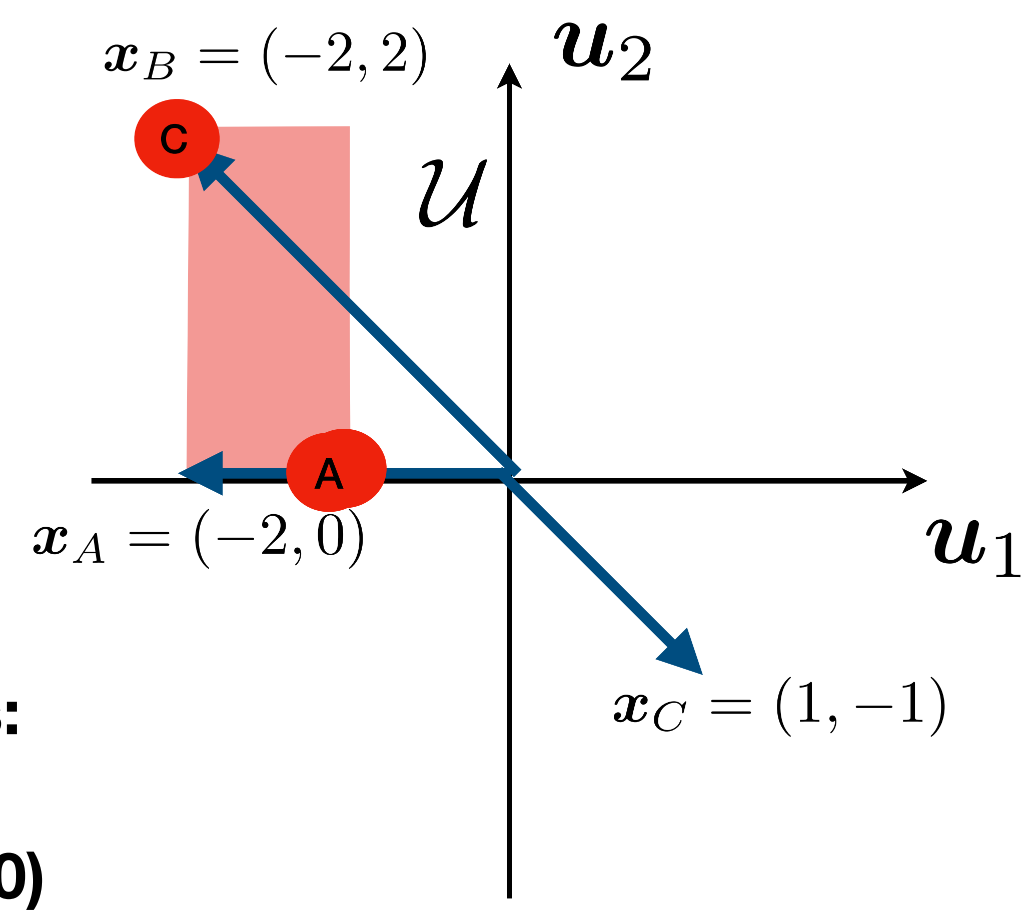

Worst-case utilities:

$\boldsymbol{x}_A$: **2 , u = (-1, 0)**

$\boldsymbol{x}_B$: **2 , u = (-1, 0)**

$\boldsymbol{x}_C$: **-4 , u = (-2, 2)**

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

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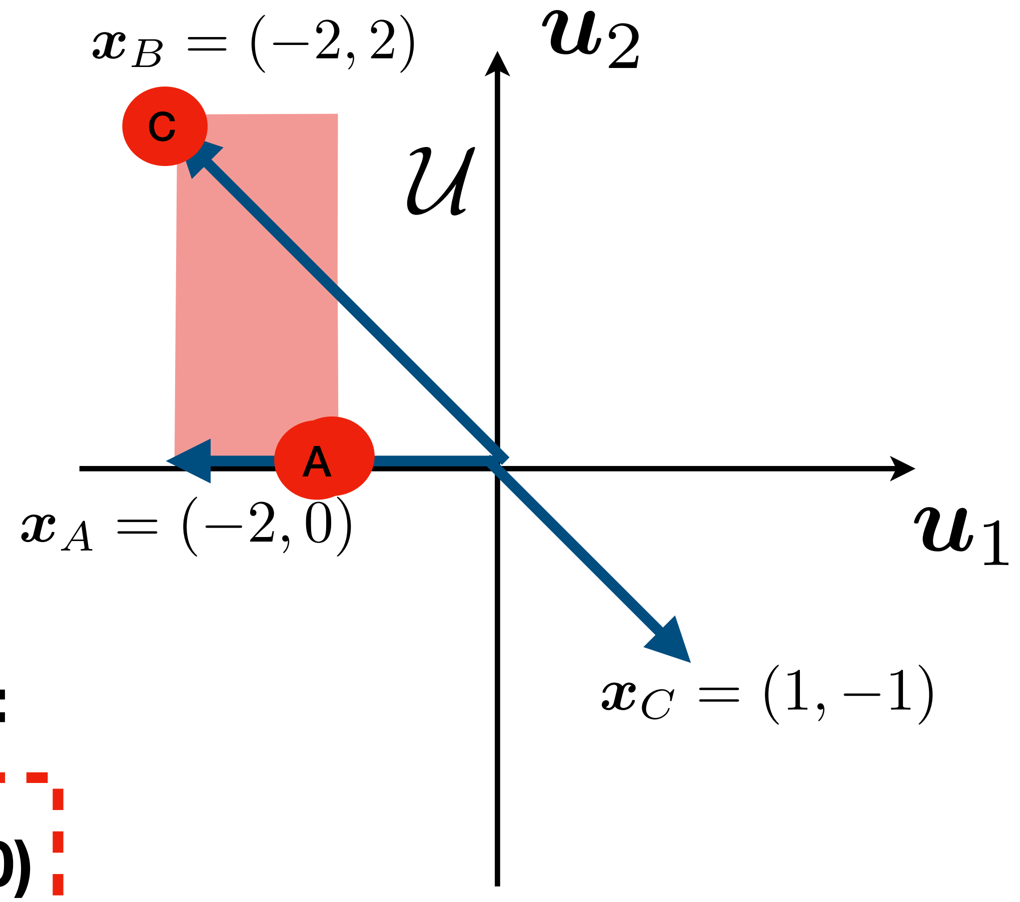

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$\boldsymbol{x}_C$: **-4 , u = (-2, 2)**

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

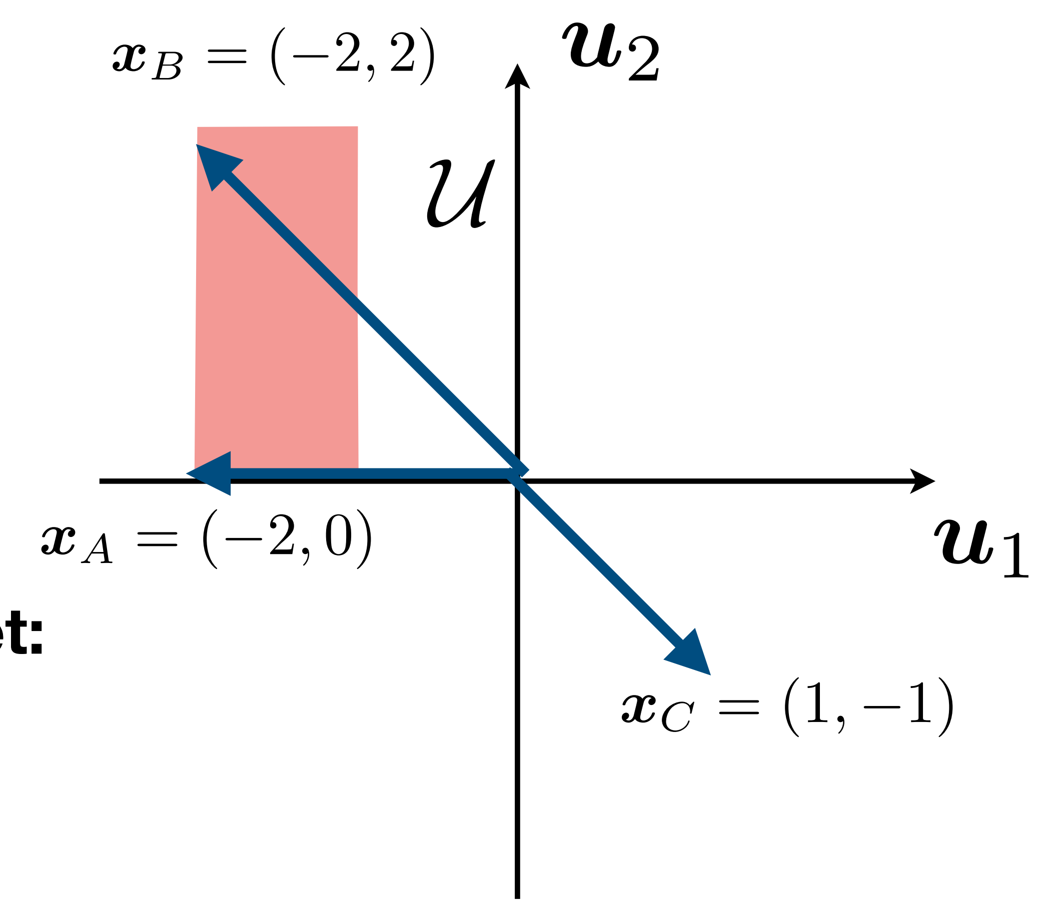
Minimax Regret

Select the item which minimizes ***worst-case regret***

Worst-case (max) regret:

$$MR(\boldsymbol{x}) \equiv \max_{\boldsymbol{u} \in \mathcal{U}} \left[\left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x} \right]$$

$$\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x} \in \mathcal{X}} MR(\boldsymbol{x})$$



Worst-case regret:

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

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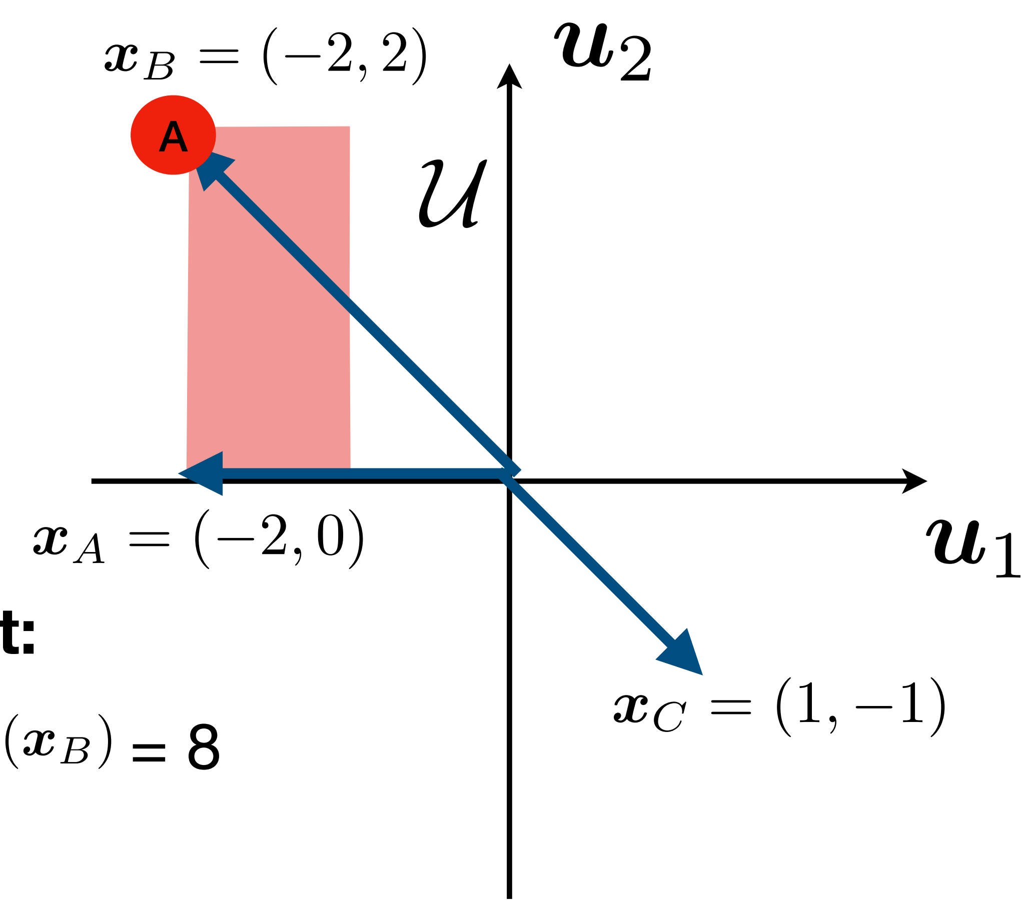


Worst-case regret:

$\boldsymbol{x}_A$: u = (-2, 2), $u(\boldsymbol{x}_B)$ = 8

***MR* = 8 - 4 = 4**

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

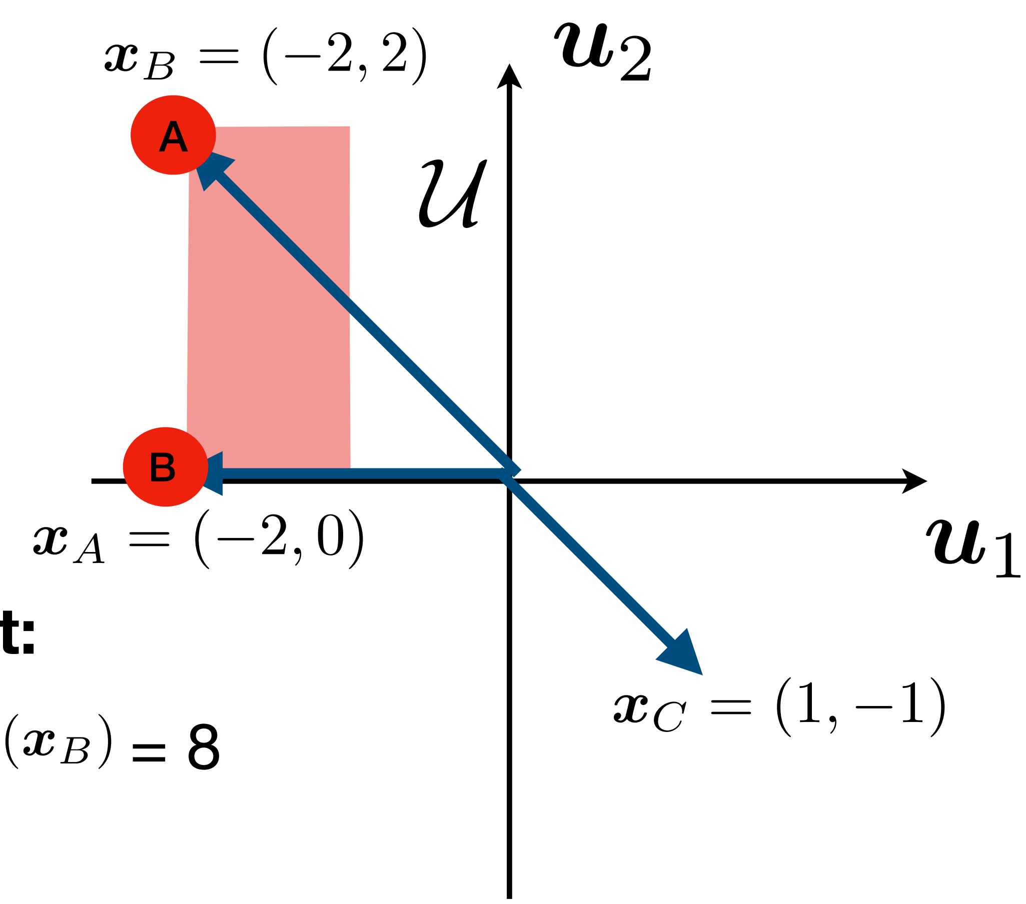
Minimax Regret

Select the item which minimizes ***worst-case regret***

Worst-case (max) regret:

$$MR(\boldsymbol{x}) \equiv \max_{\boldsymbol{u} \in \mathcal{U}} \left[\left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x} \right]$$

$$\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x} \in \mathcal{X}} MR(\boldsymbol{x})$$



Worst-case regret:

$\boldsymbol{x}_A$: u = (-2, 2), $u(\boldsymbol{x}_B)$ = 8

MR **= 8 - 4 = 4**

$\boldsymbol{x}_B$: u = (-1, 0), $u(\boldsymbol{x}_A)$ = 2

MR **= 2 - 2 = 0**

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

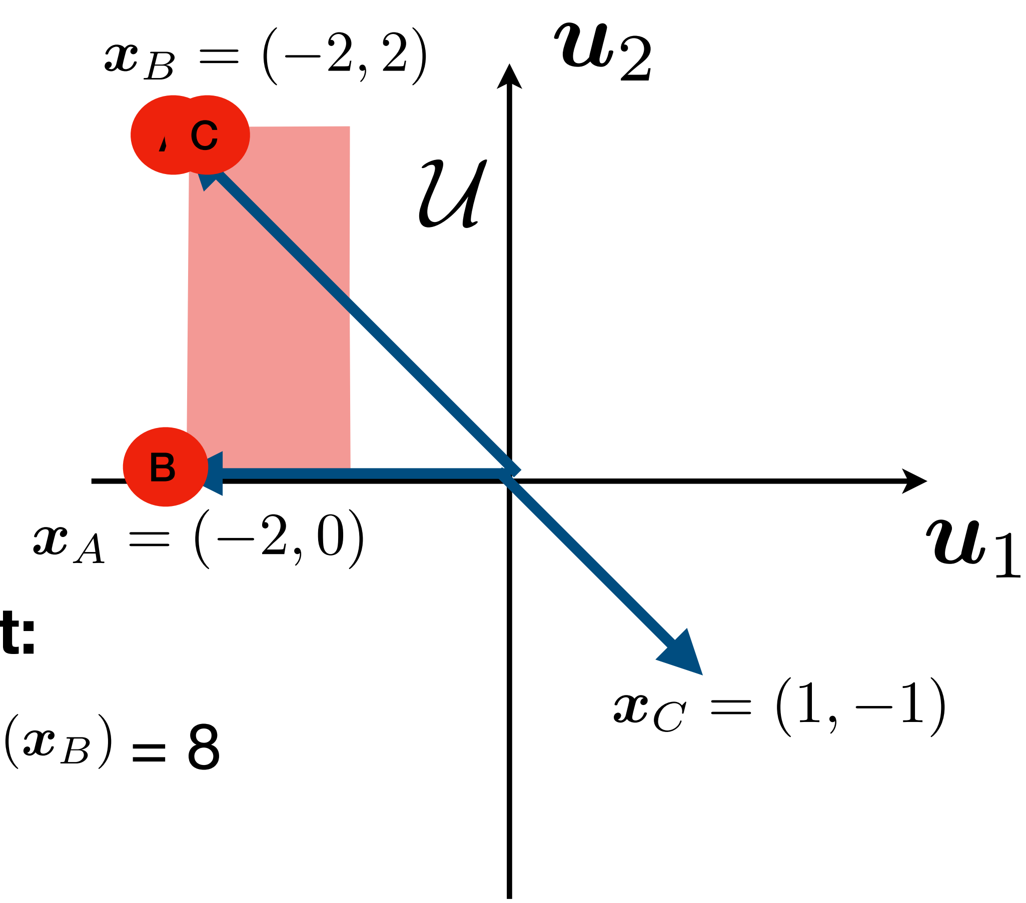
Minimax Regret

Select the item which minimizes ***worst-case regret***

Worst-case (max) regret:

$$MR(\boldsymbol{x}) \equiv \max_{\boldsymbol{u} \in \mathcal{U}} \left[\left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x} \right]$$

$$\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x} \in \mathcal{X}} MR(\boldsymbol{x})$$



Worst-case regret:

$\boldsymbol{x}_A$: u = (-2, 2), $u(\boldsymbol{x}_B)$ = 8
***MR* = 8 - 4 = 4**

$\boldsymbol{x}_B$: u = (-1, 0), $u(\boldsymbol{x}_A)$ = 2
***MR* = 2 - 2 = 0**

$\boldsymbol{x}_C$: u = (-2, 2), $u(\boldsymbol{x}_B)$ = 8
***MR* = 8 - (-4) = 12**

Recommendation

Decision Criteria: Example

$$\mathcal{U} = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R} : \boldsymbol{u}_1 \in [-2, -1], \boldsymbol{u}_2 \in [0, 2]\}$$

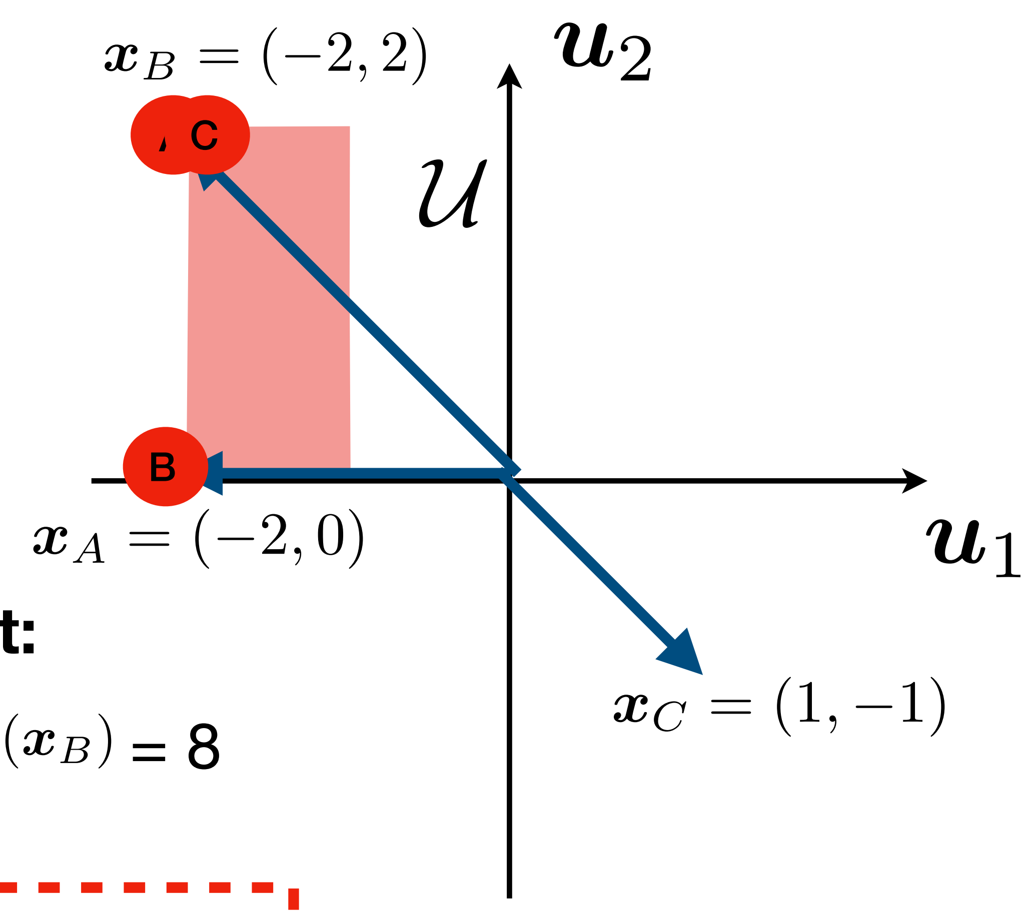
Minimax Regret

Select the item which minimizes ***worst-case regret***

Worst-case (max) regret:

$$MR(\boldsymbol{x}) \equiv \max_{\boldsymbol{u} \in \mathcal{U}} \left[\left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x} \right]$$

$$\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x} \in \mathcal{X}} MR(\boldsymbol{x})$$



Worst-case regret:

$\boldsymbol{x}_A$: u = (-2, 2), $u(\boldsymbol{x}_B)$ = 8

***MR* = 8 - 4 = 4**

$\boldsymbol{x}_B$: u = (-1, 0), $u(\boldsymbol{x}_A)$ = 2

***MR* = 2 - 2 = 0**

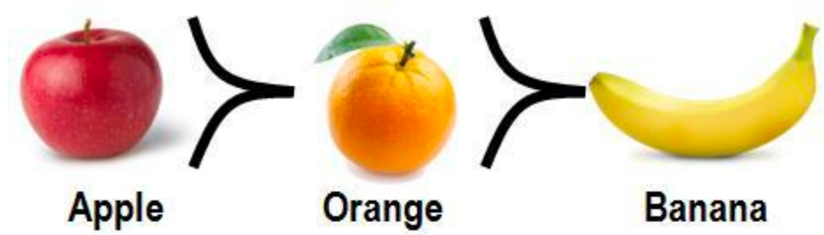
$\boldsymbol{x}_C$: u = (-2, 2), $u(\boldsymbol{x}_B)$ = 8

***MR* = 8 - (-4) = 12**

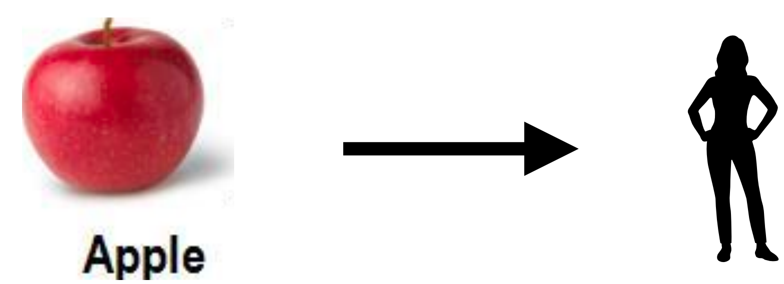
Outline

- Application: Learning an Objective Function

- Preference Elicitation



- Recommendation Under Uncertainty



- Elicitation + Recommendation

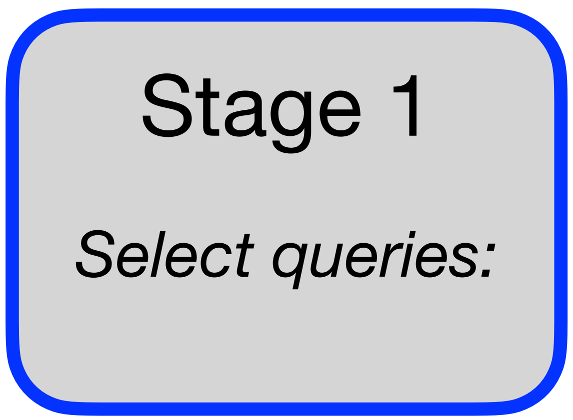
Elicitation + Recommendation

Multi-Stage Decision Process

blue = we choose
red = unknown ("nature" chooses)

Elicitation + Recommendation

Multi-Stage Decision Process



blue = we choose
red = unknown (“nature” chooses)

Elicitation + Recommendation

Multi-Stage Decision Process

Stage 1

Select queries:

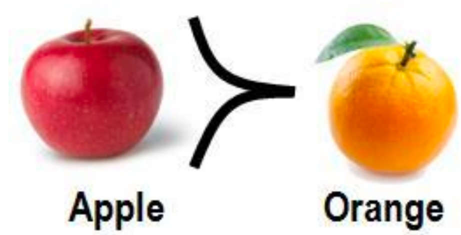
➡

Stage 1.5

Observe responses

or

?



blue = we choose
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Elicitation + Recommendation

Multi-Stage Decision Process

Stage 1

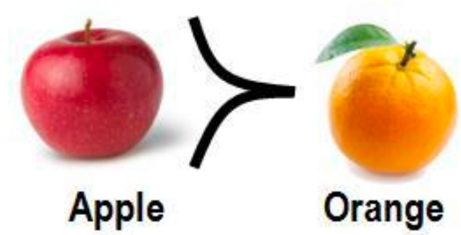
Select queries:

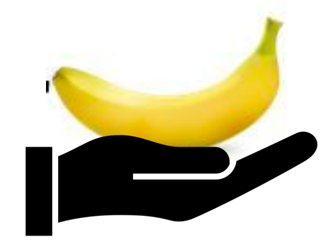
Stage 1.5

Observe responses

Stage 2

Recommend item





blue = we choose
red = unknown (“nature” chooses)

Elicitation + Recommendation

Multi-Stage Decision Process: Formalism

blue = we choose
red = unknown ("nature" chooses)

Elicitation + Recommendation

Multi-Stage Decision Process: Formalism

Stage 1

Select K queries:

$$\boldsymbol{q} \in \mathcal{Q}^K$$

Any pair of distinct items can be a query

ex, with *N* = *3* items:
$\mathcal{Q} := \{(1,2),\ (1,3),\ (2,3)\}$

blue = we choose
red = unknown ("nature" chooses)

Elicitation + Recommendation

Multi-Stage Decision Process: Formalism

Stage 1

Select K queries:

$\boldsymbol{q} \in \mathcal{Q}^K$

→

Stage 1.5

Observe responses

$\boldsymbol{s} \in \mathcal{S}^K$

Any pair of distinct items can be a query

ex, with *N* = *3* items:

$\mathcal{Q} := \{(1,2),\ (1,3),\ (2,3)\}$

Agents can respond with a strict preference, or indifference:

$$\boldsymbol{s}_\kappa = \begin{cases} 1 & \text{if } \boldsymbol{x}^{\iota_1^\kappa} \succ \boldsymbol{x}^{\iota_2^\kappa} \\ 0 & \text{if } \boldsymbol{x}^{\iota_1^\kappa} \sim \boldsymbol{x}^{\iota_2^\kappa} \\ -1 & \text{else.} \end{cases}$$

response set : $\mathcal{S} := \{-1, 0, 1\}$

blue = we choose
red = unknown ("nature" chooses)

Elicitation + Recommendation

Multi-Stage Decision Process: Formalism

Stage 1

Select K queries:

$\boldsymbol{q} \in \mathcal{Q}^K$

→

Stage 1.5

Observe responses

$\boldsymbol{s} \in \mathcal{S}^K$

→

Stage 2

Recommend item

$\boldsymbol{x} \in \mathcal{X}$

Any pair of distinct items can be a query

ex, with *N* = *3* items:

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blue = we choose

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Elicitation + Recommendation

Multi-Stage Decision Process: Formalism

Stage 1

Select K queries:

$\boldsymbol{q} \in \mathcal{Q}^K$

➡

Stage 1.5

Observe responses

$\boldsymbol{s} \in \mathcal{S}^K$

➡

Stage 2

Recommend item

$\boldsymbol{x} \in \mathcal{X}$

Any pair of distinct items can be a query

ex, with *N* = *3* items:

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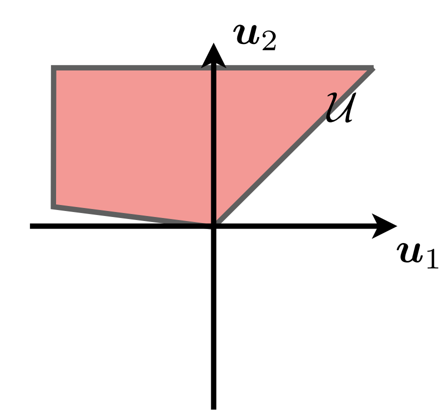
response set : $\mathcal{S} := \{-1, 0, 1\}$

Note: Final uncertainty set depends on both prior **queries** and **responses!**



blue = we choose

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Elicitation + Recommendation

Multi-Stage Decision Process: Formalism

Stage 1

Select K queries:

$\boldsymbol{q} \in \mathcal{Q}^K$

➡

Stage 1.5

Observe responses

$\boldsymbol{s} \in \mathcal{S}^K$

➡

Stage 2

Recommend item

$\boldsymbol{x} \in \mathcal{X}$

Comment

Stage 1 & Stage 2 are linked through the *agent uncertainty set*

$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \begin{array}{lllll} \boldsymbol{u} \in \mathcal{U}^0 & : & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) > 0 & \forall \kappa = 1, \ldots, K : & \boldsymbol{s}_\kappa = 1 \\ & & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) = 0 & \forall \kappa = 1, \ldots, K : & \boldsymbol{s}_\kappa = 0 \\ & & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) < 0 & \forall \kappa = 1, \ldots, K : & \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

blue = we choose

red = unknown ("nature" chooses)

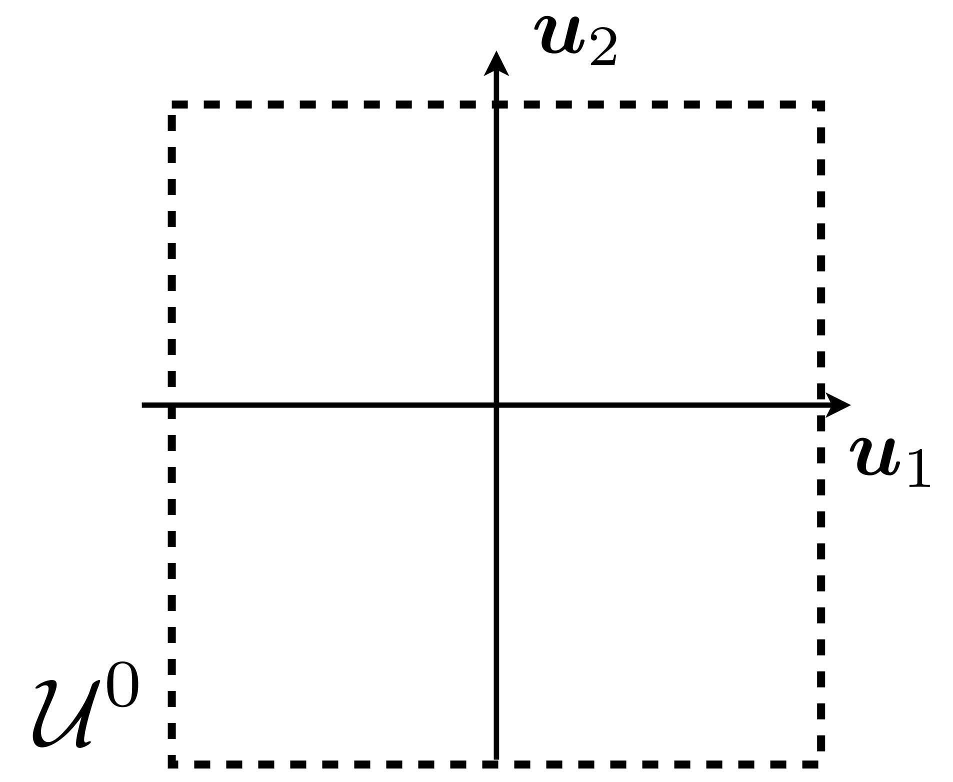
Elicitation + Recommendation

Agent Uncertainty Set

Comment

Stage 1 & Stage 2 are linked through the *agent uncertainty set*

$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \begin{array}{lll} \boldsymbol{u} \in \mathcal{U}^0 & : & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) > 0 \quad \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = 1 \\ & & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) = 0 \quad \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = 0 \\ & & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) < 0 \quad \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$



(initial uncertainty set)

blue = we choose
red = unknown (“nature” chooses)

Elicitation + Recommendation

Agent Uncertainty Set

Comment

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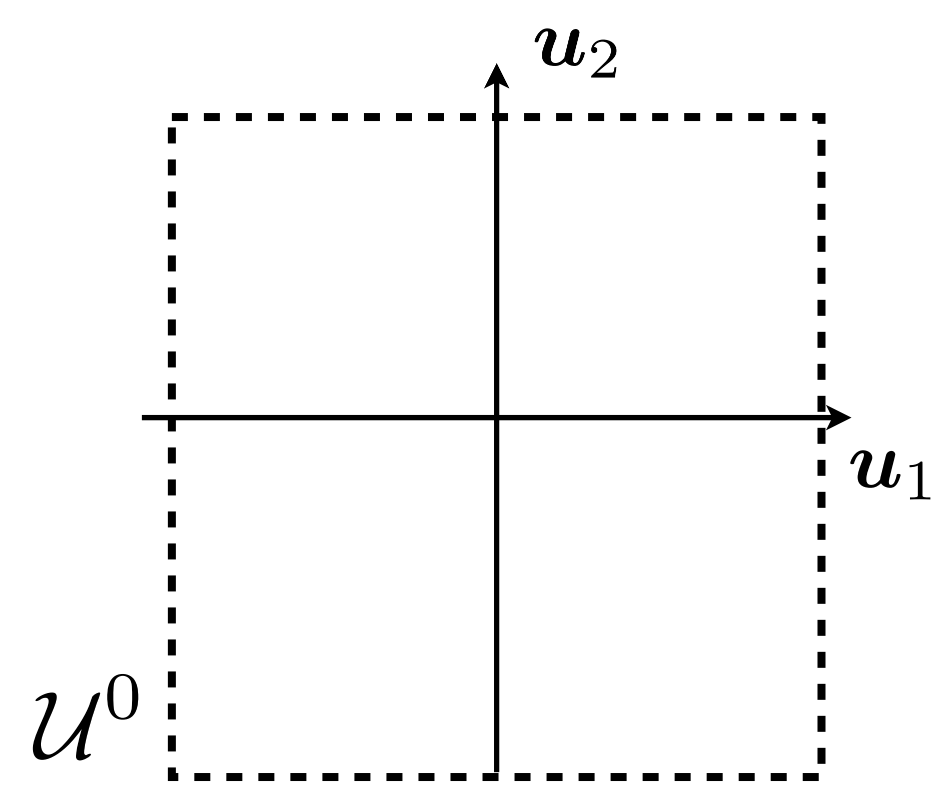
$$\boldsymbol{x}^{\boldsymbol{q}_1^A} - \boldsymbol{x}^{\boldsymbol{q}_1^B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{s}_1 = 1$$



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Elicitation + Recommendation

Agent Uncertainty Set

Comment

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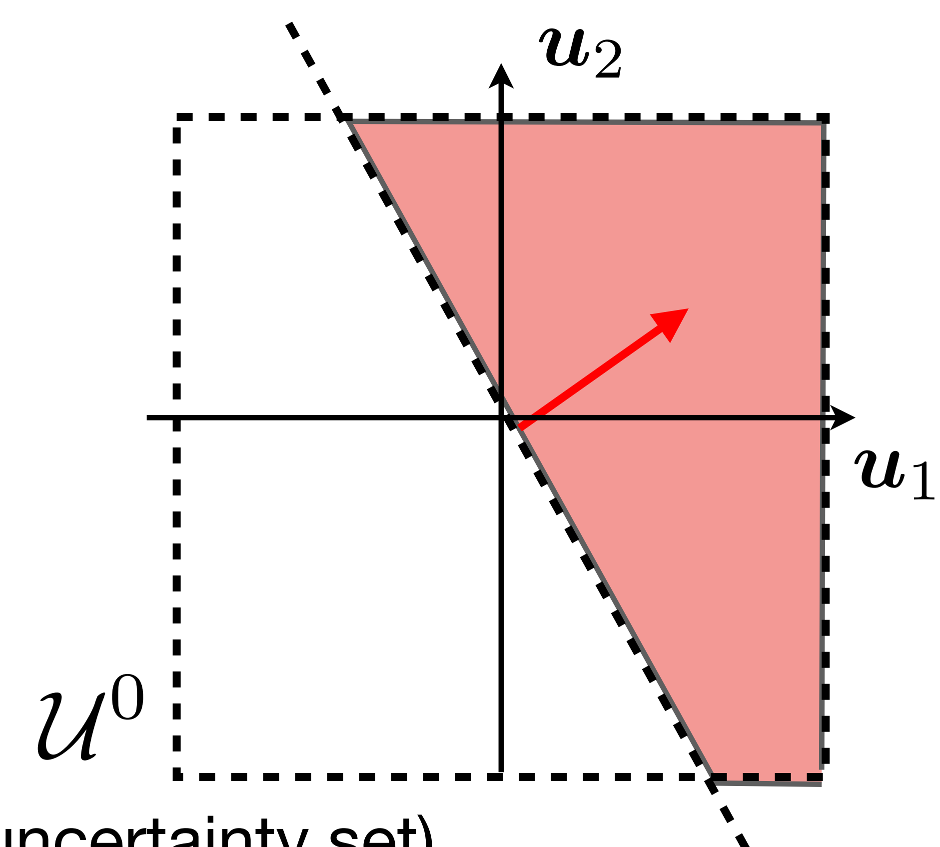
$$\boldsymbol{x}^{\boldsymbol{q}_1^A} - \boldsymbol{x}^{\boldsymbol{q}_1^B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{s}_1 = 1 \quad \Longrightarrow \quad 2\boldsymbol{u}_1 + \boldsymbol{u}_2 > 0$$



(initial uncertainty set)

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Elicitation + Recommendation

Agent Uncertainty Set

Comment

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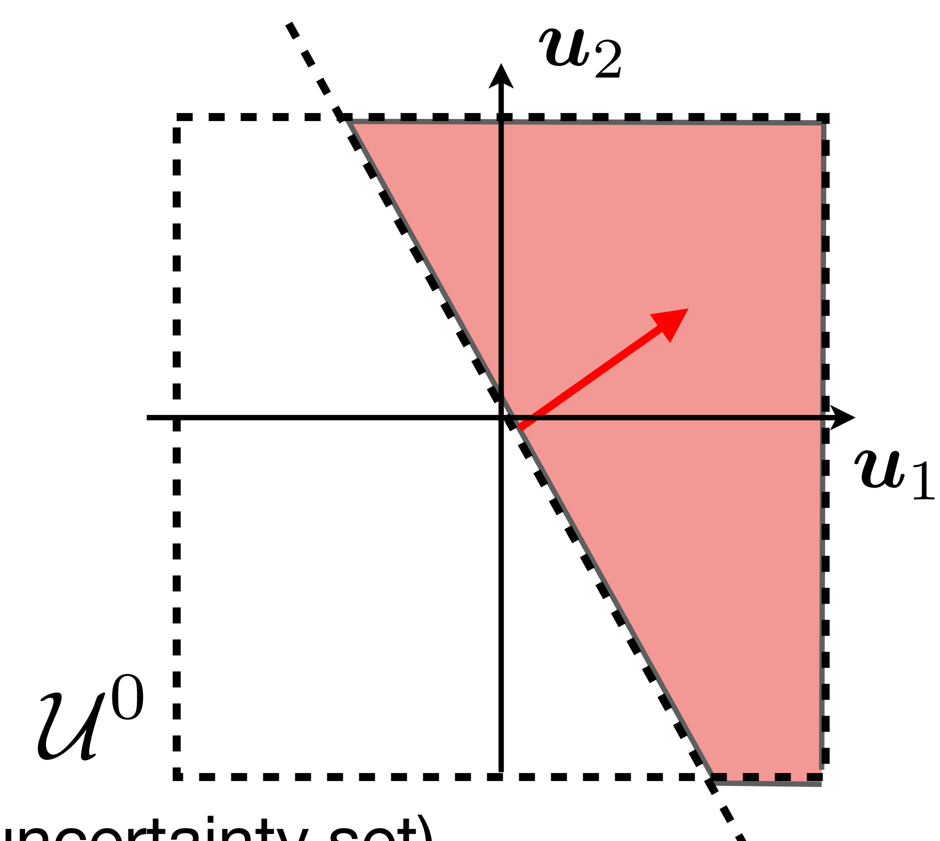
$$\boldsymbol{x}^{\boldsymbol{q}_1^A} - \boldsymbol{x}^{\boldsymbol{q}_1^B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{s}_1 = 1 \quad \Longrightarrow \quad 2\boldsymbol{u}_1 + \boldsymbol{u}_2 > 0$$

$$\boldsymbol{x}^{\boldsymbol{q}_2^A} - \boldsymbol{x}^{\boldsymbol{q}_2^B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{s}_2 = -1$$



(initial uncertainty set)

blue = we choose
red = unknown (“nature” chooses)

Elicitation + Recommendation

Agent Uncertainty Set

Comment

Stage 1 & Stage 2 are linked through the *agent uncertainty set*

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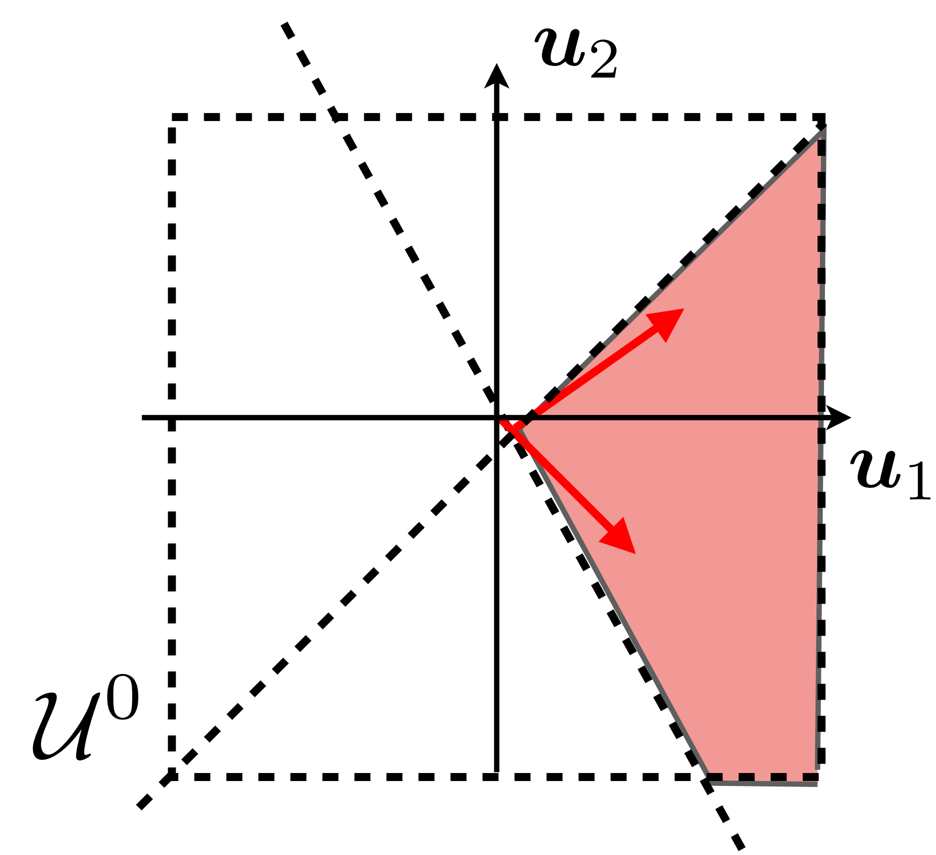
$$\boldsymbol{x}^{\boldsymbol{q}_1^A} - \boldsymbol{x}^{\boldsymbol{q}_1^B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{s}_1 = 1 \implies 2\boldsymbol{u}_1 + \boldsymbol{u}_2 > 0$$

$$\boldsymbol{x}^{\boldsymbol{q}_2^A} - \boldsymbol{x}^{\boldsymbol{q}_2^B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{s}_2 = -1 \implies -\boldsymbol{u}_1 + \boldsymbol{u}_2 < 0$$



blue = we choose
red = unknown (“nature” chooses)

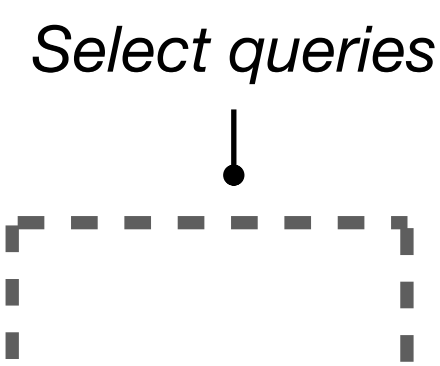
Elicitation + Recommendation

Formalizing an Optimization Problem (MMU)

Elicitation + Recommendation

Formalizing an Optimization Problem (MMU)

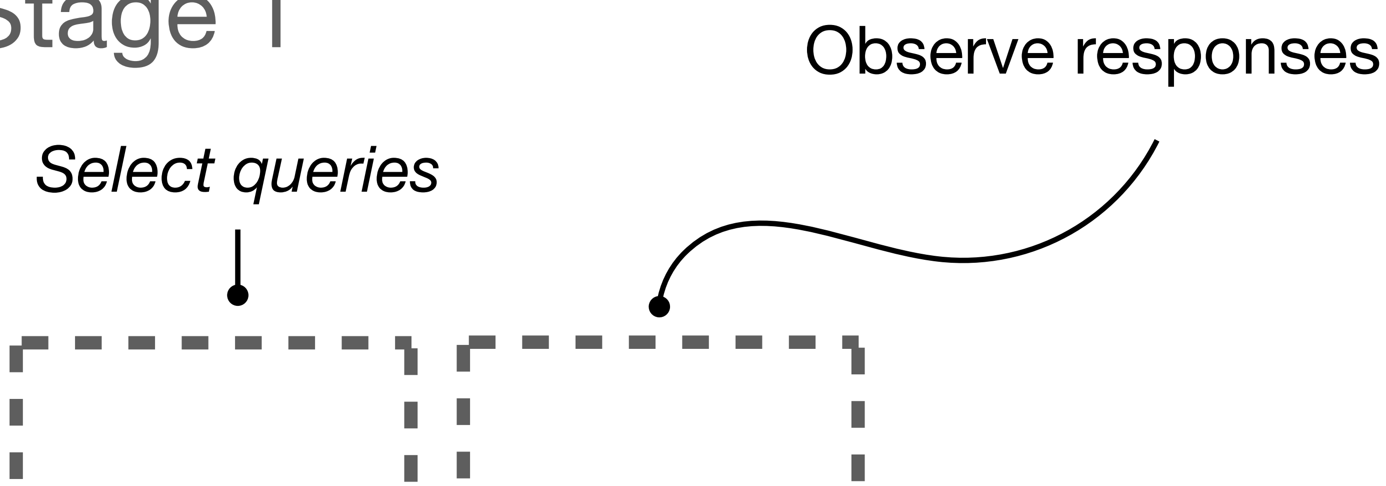
Stage 1



Elicitation + Recommendation

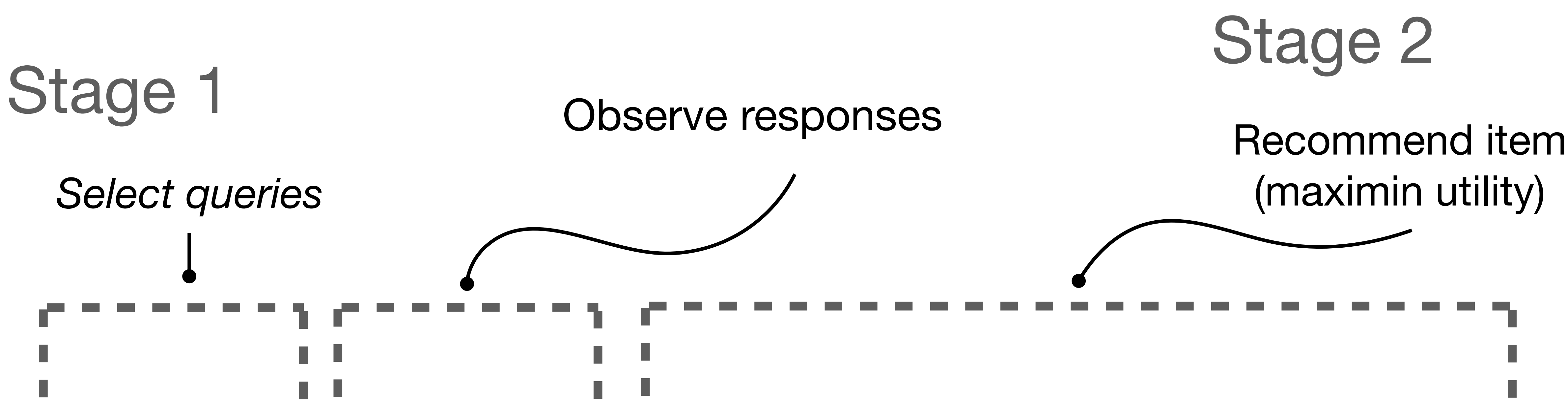
Formalizing an Optimization Problem (MMU)

Stage 1



Elicitation + Recommendation

Formalizing an Optimization Problem (MMU)



Elicitation + Recommendation

Formalizing an Optimization Problem (MMU)

Stage 1

Stage 2

Select queries

Observe responses

Recommend item
(maximin utility)

$$\max_{\boldsymbol{x} \in \mathcal{X}} \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}$$

Elicitation + Recommendation

Formalizing an Optimization Problem (MMU)

Stage 1

Stage 2

Select queries

Observe responses

Recommend item
(maximin utility)

$$\min_{\boldsymbol{s}\in\mathcal{S}^K} \max_{\boldsymbol{x}\in\mathcal{X}} \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}$$

"bad" = min

Elicitation + Recommendation

Formalizing an Optimization Problem (MMU)

Stage 1

Stage 2

Select queries

Observe responses

Recommend item (maximin utility)

$$\min_{\boldsymbol{q}\in\mathcal{Q}^K}\ \min_{\boldsymbol{s}\in\mathcal{S}^K}\ \max_{\boldsymbol{x}\in\mathcal{X}}\ \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})}\ \boldsymbol{u}^\top\boldsymbol{x}$$

"good" = max "bad" = min

Elicitation + Recommendation

Formalizing an Optimization Problem (MMU)

Stage 1

Stage 2

Select queries

Observe responses

Recommend item (maximin utility)

$$\min_{\boldsymbol{q} \in \mathcal{Q}^K} \; \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \max_{\boldsymbol{x} \in \mathcal{X}} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}$$

Comment (1)

"Robust" Approach: Assume that "nature" selects the **worst possible outcomes:**

- Over responses $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})$
- Over agent utility $\boldsymbol{s} \in \mathcal{S}^K$

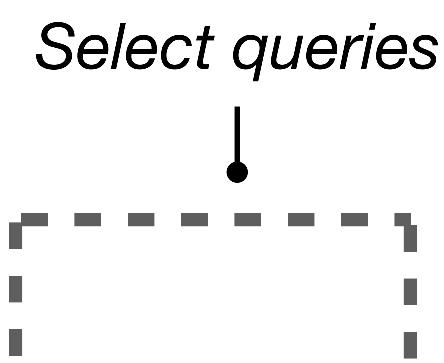
Elicitation + Recommendation

Formalizing an Optimization Problem (MMR)

Elicitation + Recommendation

Formalizing an Optimization Problem (MMR)

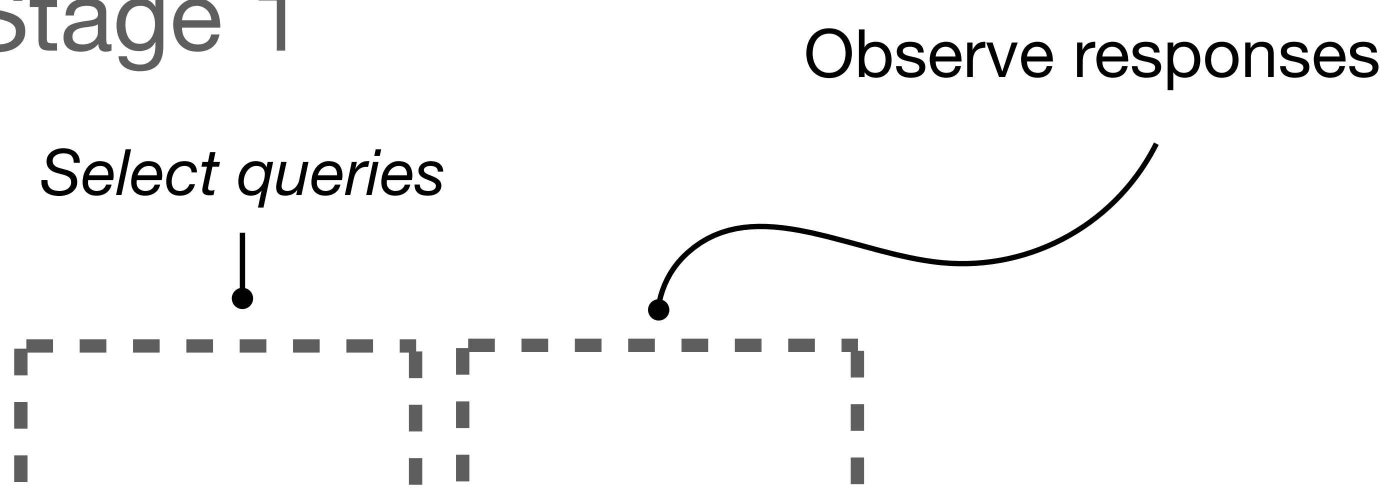
Stage 1



Elicitation + Recommendation

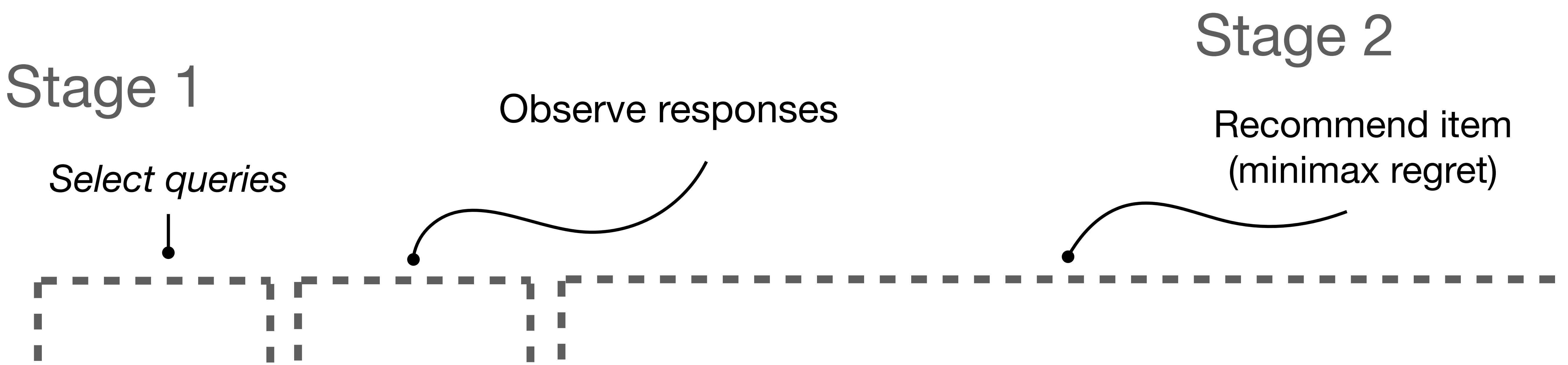
Formalizing an Optimization Problem (MMR)

Stage 1



Elicitation + Recommendation

Formalizing an Optimization Problem (MMR)



Elicitation + Recommendation

Formalizing an Optimization Problem (MMR)

Stage 1

Select queries

Observe responses

Stage 2

Recommend item (minimax regret)

$$\min_{\boldsymbol{x} \in \mathcal{X}} \max_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \left[\left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x} \right]$$

Elicitation + Recommendation

Formalizing an Optimization Problem (MMR)

Stage 1

Stage 2

Select queries

Observe responses

Recommend item
(minimax regret)

$$\max_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{x} \in \mathcal{X}} \max_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \left[\left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x} \right]$$

"bad" = max

Elicitation + Recommendation

Formalizing an Optimization Problem (MMR)

Stage 1

Stage 2

Select queries

Observe responses

Recommend item
(minimax regret)

$$\min_{\boldsymbol{q}\in\mathcal{Q}^K} \; \max_{\boldsymbol{s}\in\mathcal{S}^K} \; \min_{\boldsymbol{x}\in\mathcal{X}} \; \max_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \left[\left(\max_{\boldsymbol{x}'\in\mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x} \right]$$

“good” = min “bad” = max

Elicitation + Recommendation

Formalizing an Optimization Problem (MMR)

Stage 1

Select queries

Observe responses

Stage 2

Recommend item
(minimax regret)

$$\min_{\boldsymbol{q} \in \mathcal{Q}^K} \; \max_{\boldsymbol{s} \in \mathcal{S}^K} \; \min_{\boldsymbol{x} \in \mathcal{X}} \; \max_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \left[\left(\max_{\boldsymbol{x}' \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}' \right) - \boldsymbol{u}^\top \boldsymbol{x} \right]$$

“good” = min “bad” = max

This is a huge optimization problem…
How do we solve it?

(next time?)