Preference Elicitation & Recommendation

Part 2

Applied Mechanism Design for Social Good — CMSC828M
23 April, 2020

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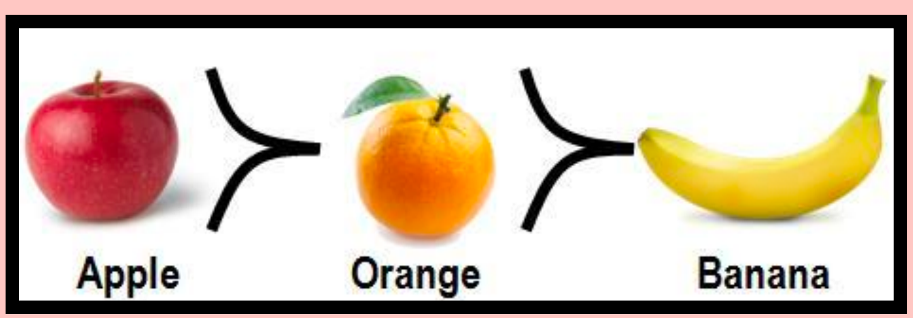
Elicitation + Recommendation

Preference Elicitation

What does the {agent | customer | user} want?

Select queries *Observe responses*

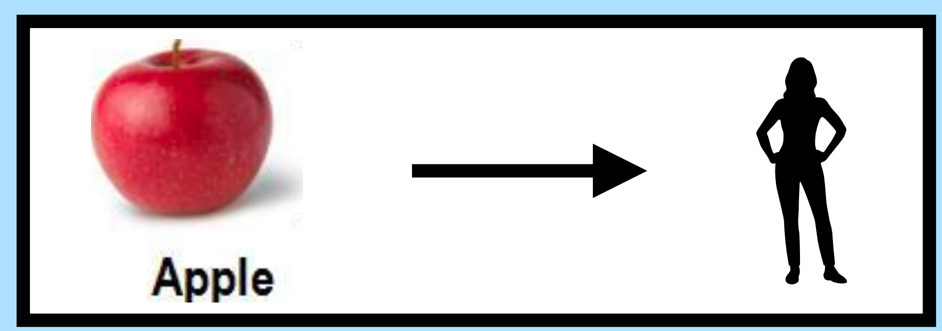
$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \min_{\boldsymbol{s} \in \mathcal{S}^K}$$



Recommendation

Which item should we offer?

$$\boldsymbol{x}^* \in \arg\max_{\boldsymbol{x} \in \mathcal{X}} \min_{\boldsymbol{u} \in \mathcal{U}} \; \boldsymbol{u}^\top \boldsymbol{x}$$



Elicitation + Recommendation

Multi-Stage Optimization

Stage 1

Stage 2

Select queries

Observe responses

Recommend item (maximin utility)

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \; \min_{\boldsymbol{s}\in\mathcal{S}^K} \; \max_{\boldsymbol{x}\in\mathcal{X}} \; \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}$$

Elicitation + Recommendation

Multi-Stage Optimization

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Elicitation + Recommendation

Multi-Stage Optimization

Sequence of decisions

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \quad \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

Elicitation + Recommendation

Multi-Stage Optimization

Sequence of decisions

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \left\{ \begin{array}{l} \min\limits_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max\limits_{\boldsymbol{x} \in \mathcal{X}} \quad \min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x} \\ \\ \text{s.t.} \quad \boldsymbol{q} \quad \text{is fixed} \end{array} \right\}$$

Elicitation + Recommendation

Multi-Stage Optimization

Sequence of decisions

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \left\{ \begin{array}{l} \displaystyle \min_{\boldsymbol{s} \in \mathcal{S}^K} \left\{ \begin{array}{l} \displaystyle \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x} \\ \\ \text{s.t.} \quad \boldsymbol{s}, \boldsymbol{q} \quad \text{are fixed} \end{array} \right\} \\ \text{s.t.} \quad \boldsymbol{q} \quad \text{is fixed} \end{array} \right\}$$

Elicitation + Recommendation

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Sequence of decisions

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Elicitation + Recommendation

Comment

$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \begin{array}{lll} \boldsymbol{u} \in \mathcal{U}^0 & : & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}^A_\kappa} - \boldsymbol{x}^{\boldsymbol{q}^B_\kappa}) > 0 \quad \forall \kappa = 1, \ldots, K : \quad \boldsymbol{s}_\kappa = 1 \\ & & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}^A_\kappa} - \boldsymbol{x}^{\boldsymbol{q}^B_\kappa}) = 0 \quad \forall \kappa = 1, \ldots, K : \quad \boldsymbol{s}_\kappa = 0 \\ & & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}^A_\kappa} - \boldsymbol{x}^{\boldsymbol{q}^B_\kappa}) < 0 \quad \forall \kappa = 1, \ldots, K : \quad \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

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Indifference response does not impact our optimization problems

Elicitation + Recommendation

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Elicitation + Recommendation

Stage 1

Stage 2

Select queries

Observe responses

Recommend item
(maximin utility)

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Now, some theory

Elicitation + Recommendation

Observation

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Suppose we can "design" an item

$$\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^J \mid \boldsymbol{x}^\top \boldsymbol{b} \leq \boldsymbol{c}\}$$

Elicitation + Recommendation

Observation

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Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

Elicitation + Recommendation

Observation (Step 1)

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Minimax Theorem: (von Neumann)

- If $f(\boldsymbol{x}, \boldsymbol{y})$ is convex in ***x*****,** concave in ***y***
- On convex compact sets ***X*** and ***Y***, then:

$$\max_{\boldsymbol{x} \in \mathcal{X}} \min_{\boldsymbol{y} \in \mathcal{Y}} f(\boldsymbol{x}, \boldsymbol{y}) = \min_{\boldsymbol{y} \in \mathcal{Y}} \max_{\boldsymbol{x} \in \mathcal{X}} f(\boldsymbol{x}, \boldsymbol{y})$$

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Elicitation + Recommendation

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Elicitation + Recommendation

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Convex/concave
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Elicitation + Recommendation

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$$\min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})}$$

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Elicitation + Recommendation

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- If $f(\boldsymbol{x}, \boldsymbol{y})$ is convex in ***x*,** concave in ***y*** ✓
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Elicitation + Recommendation

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Elicitation + Recommendation

Observation (Step 2)

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Elicitation + Recommendation

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Minimizing both $\boldsymbol{s}\in\mathcal{S}^K$ and $\boldsymbol{u}$ **simultaneously** means "nature" can choose **any** $\boldsymbol{u}$

Elicitation + Recommendation

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Elicitation + Recommendation

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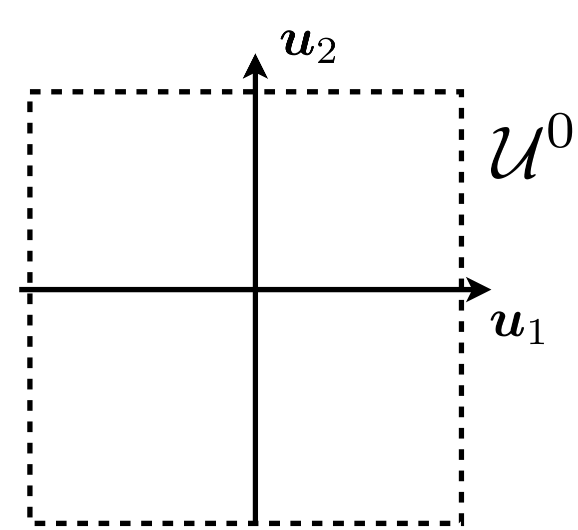

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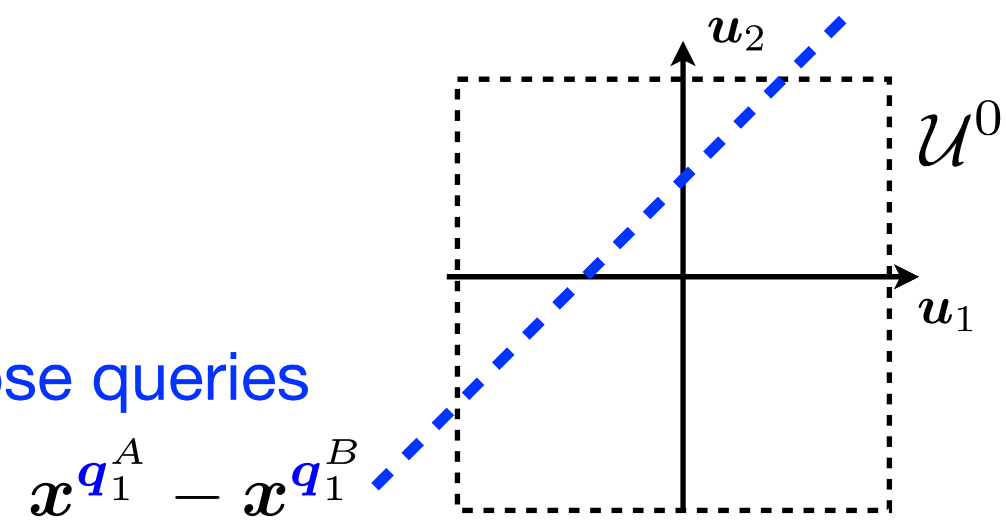
With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

(1)

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

1) we choose queries



$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \boldsymbol{u} \in \mathcal{U}^0 \; : \; \begin{array}{ll} \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \geq 0 & \forall \kappa = 1, \ldots, K : \;\; \boldsymbol{s}_\kappa = 1 \\ \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \leq 0 & \forall \kappa = 1, \ldots, K : \;\; \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

Elicitation + Recommendation

Observation (Step 2)

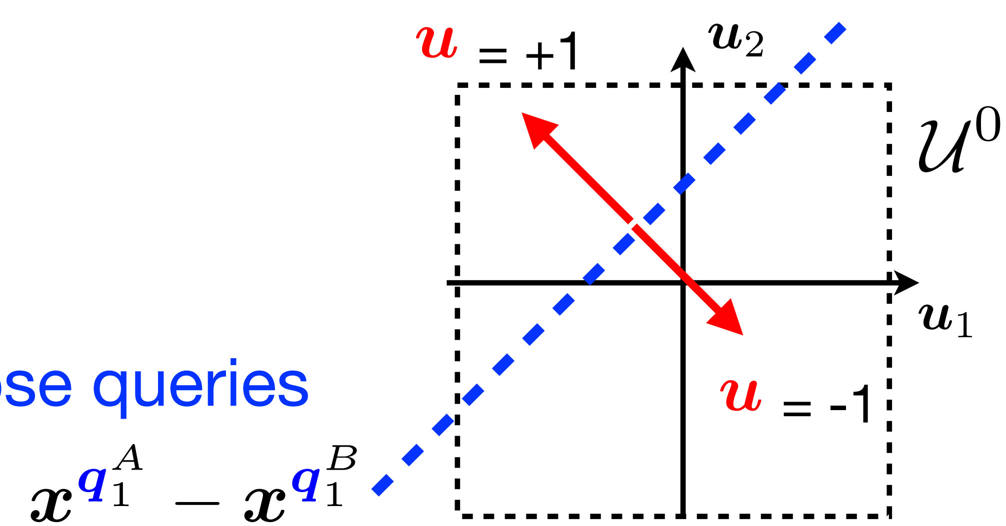
Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

(1) (2)

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \ \min_{\boldsymbol{s} \in \mathcal{S}^K} \ \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \ \max_{\boldsymbol{x} \in \mathcal{X}} \ \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)



2) nature chooses responses

1) we choose queries

$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \begin{array}{lll} \boldsymbol{u} \in \mathcal{U}^0 & : & \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \geq 0 \quad \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = 1 \\ & & \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \leq 0 \quad \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

Elicitation + Recommendation

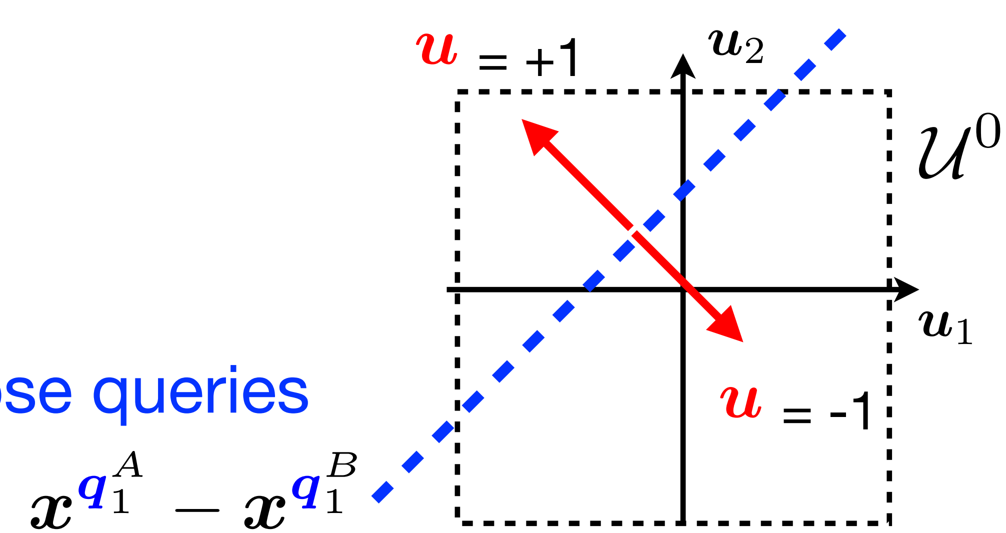
Observation (Step 2)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\underset{(1)}{\max_{\boldsymbol{q} \in \mathcal{Q}^K}} \; \underset{(2)}{\min_{\boldsymbol{s} \in \mathcal{S}^K}} \; \underset{(3)}{\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})}} \; \max_{\boldsymbol{x} \in \mathcal{X}} \; \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)



1) we choose queries

2) nature chooses responses

3) nature chooses $\boldsymbol{u}$

$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \boldsymbol{u} \in \mathcal{U}^0 \;:\; \begin{array}{ll} \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \geq 0 & \forall \kappa = 1, \ldots, K : \;\; \boldsymbol{s}_\kappa = 1 \\ \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \leq 0 & \forall \kappa = 1, \ldots, K : \;\; \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

Elicitation + Recommendation

Observation (Step 2)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \begin{array}{lll} \boldsymbol{u} \in \mathcal{U}^0 \ : & \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \geq 0 & \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = 1 \\ & \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \leq 0 & \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

Elicitation + Recommendation

Observation (Step 2)

Observation

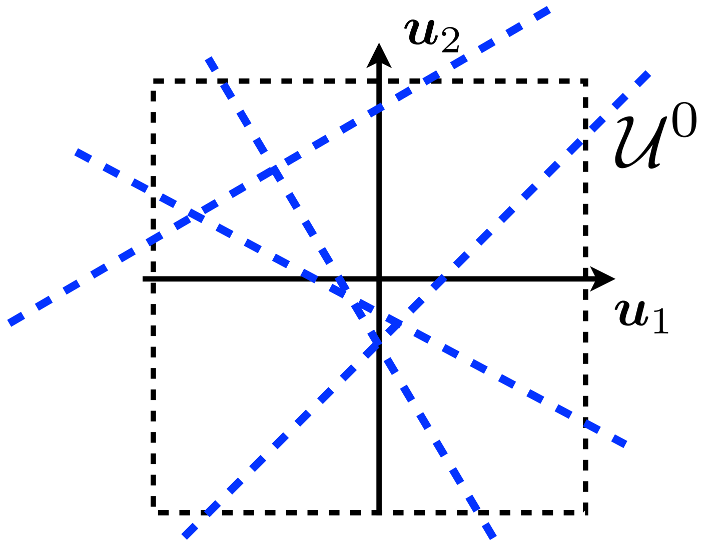
With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

Q: Suppose I ask several queries:

Does this restrict nature from choosing any $\boldsymbol{u} \in \mathcal{U}^0$?



$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \begin{array}{lllll} \boldsymbol{u} \in \mathcal{U}^0 & : & \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \geq 0 & \forall \kappa = 1, \ldots, K : & \boldsymbol{s}_\kappa = 1 \\ & & \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \leq 0 & \forall \kappa = 1, \ldots, K : & \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

Elicitation + Recommendation

Observation (Step 2)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

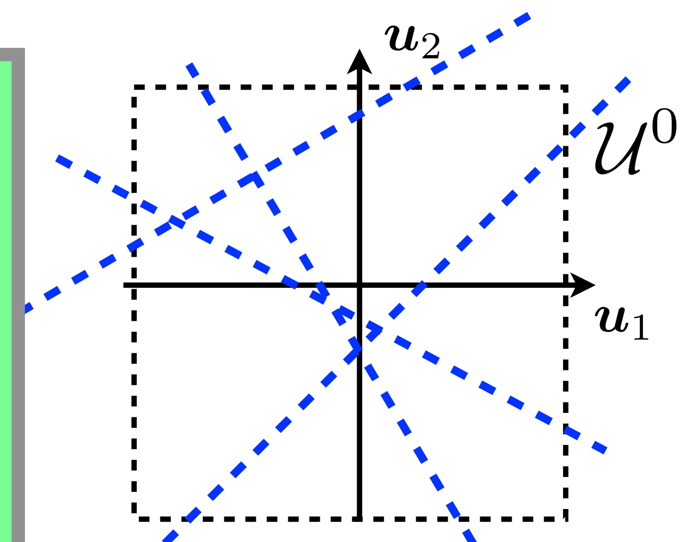
$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \min_{\boldsymbol{s} \in \mathcal{S}^K} \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \max_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

A: No, nature can select any

$$\min_{\boldsymbol{u} \in \bigcup_{\boldsymbol{s} \in \mathcal{S}^K} \mathcal{U}(\boldsymbol{q},\boldsymbol{s})}$$

Which is equivalent to any $\boldsymbol{u} \in \mathcal{U}^0$



$$\mathcal{U}(\boldsymbol{q},\boldsymbol{s}) := \left\{ \begin{array}{lll} \boldsymbol{u} \in \mathcal{U}^0 & : & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \geq 0 \quad \forall \kappa = 1, \ldots, K : \quad \boldsymbol{s}_\kappa = 1 \\ & & \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \leq 0 \quad \forall \kappa = 1, \ldots, K : \quad \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

Elicitation + Recommendation

Observation (Step 2)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \underbrace{\min_{\boldsymbol{s}\in\mathcal{S}^K} \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})}}_{\min_{\boldsymbol{u}\in\mathcal{U}^0}} \max_{\boldsymbol{x}\in\mathcal{X}} \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

(Unrestricted $\boldsymbol{u}$)

Elicitation + Recommendation

Observation (Step 3)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \quad \min_{\boldsymbol{u}\in\mathcal{U}^0} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

(Unrestricted $\boldsymbol{u}$)

Elicitation + Recommendation

Observation (Step 3)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\cancel{\max_{\boldsymbol{q}\in\mathcal{Q}^K}} \quad \min_{\boldsymbol{u}\in\mathcal{U}^0} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

(Unrestricted $\boldsymbol{u}$)

Queries don't matter

Elicitation + Recommendation

Observation (Step 3)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\cancel{\max_{\boldsymbol{q}\in\mathcal{Q}^K}} \quad \min_{\boldsymbol{u}\in\mathcal{U}^0} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

(Unrestricted $\boldsymbol{u}$)

Queries don't matter

Q: What does this mean?

Elicitation + Recommendation

Observation (Step 3)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\cancel{\max_{\boldsymbol{q}\in\mathcal{Q}^K}} \quad \min_{\boldsymbol{u}\in\mathcal{U}^0} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

(Unrestricted $\boldsymbol{u}$)

Queries don't matter

Q: What does this mean?

- don't use convex $\mathcal{X}$?

Elicitation + Recommendation

Observation (Step 3)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\cancel{\max_{\boldsymbol{q} \in \mathcal{Q}^K}} \quad \min_{\boldsymbol{u} \in \mathcal{U}^0} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

(Unrestricted $\boldsymbol{u}$)

Queries don't matter

Q: What does this mean?

- don't use convex $\mathcal{X}$?
- robustness is too conservative?

Elicitation + Recommendation

Observation (Step 3)

Observation

With maximin-utility recommendation, and a convex item set $\mathcal{X}$, **elicitation is useless.**

$$\cancel{\max_{\boldsymbol{q}\in\mathcal{Q}^K}} \quad \min_{\boldsymbol{u}\in\mathcal{U}^0} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

(Minimax Thm.)

(Unrestricted $\boldsymbol{u}$)

Queries don't matter

$$\min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})}$$

Q: What does this mean?

- don't use convex $\mathcal{X}$?
- robustness is too conservative?

Elicitation + Recommendation

MILP Reformulation

Stage 1

Stage 2

Observe responses

Recommend item
(maximin utility)

Select queries

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \; \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \max_{\boldsymbol{x} \in \mathcal{X}} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}$$

How do we solve
this problem?

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

Goal:

Express this problem as a finite-size **linear program** (with integer and/or continuous variables)

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \quad \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

Goal:

Express this problem as a finite-size **linear program** (with integer and/or continuous variables)

Reformulation Tricks

1. "Indexing" discrete variables
2. Epigraph formulation
3. Linearization
4. Duality
5. Decomposition

Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

Fixed

Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

Fixed

Suppose we *don't wait for nature* to select $\boldsymbol{s}$!

Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

Fixed

Suppose we *don't wait for nature* to select $\boldsymbol{s}$!

Instead, for each possible agent response scenario $\boldsymbol{s}$ …

Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\underbrace{\max_{\boldsymbol{q} \in \mathcal{Q}^K}}_{\text{Fixed}} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

Suppose we *don't wait for nature* to select $\boldsymbol{s}$!

Instead, for each possible agent response scenario $\boldsymbol{s}$ …

we find the optimal item to recommend, $\boldsymbol{x}^{\boldsymbol{s}}$…

Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\underbrace{\max_{\boldsymbol{q}\in\mathcal{Q}^K}}_{\text{Fixed}} \; \min_{\boldsymbol{s}\in\mathcal{S}^K} \; \cancel{\max_{\boldsymbol{x}\in\mathcal{X}}} \; \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}$$

Suppose we *don't wait for nature* to select $\boldsymbol{s}$!

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Elicitation-Recommendation Problem

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$$\underbrace{\max_{\boldsymbol{q} \in \mathcal{Q}^K}}_{\text{Fixed}} \; \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \cancel{\max_{\boldsymbol{x} \in \mathcal{X}}} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\left\{ \begin{array}{c} \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_1} \in \mathcal{X}} \\ \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\}$$

Suppose we *don't wait for nature* to select $\boldsymbol{s}$!

Instead, for each possible agent response scenario $\boldsymbol{s}$ …

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Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\underbrace{\max_{\boldsymbol{q} \in \mathcal{Q}^K}}_{\text{Fixed}} \; \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \cancel{\max_{\boldsymbol{x} \in \mathcal{X}}} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \cancel{\boldsymbol{u}^\top \boldsymbol{x}}$$

$$\left\{ \begin{array}{c} \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_1} \in \mathcal{X}} \\ \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \qquad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Suppose we *don't wait for nature* to select $\boldsymbol{s}$!

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Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\underbrace{\max_{\boldsymbol{q} \in \mathcal{Q}^K}}_{\text{Fixed}} \; \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \cancel{\max_{\boldsymbol{x} \in \mathcal{X}}} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \cancel{\boldsymbol{u}^\top \boldsymbol{x}}$$

$$\left\{ \begin{matrix} \max_{\boldsymbol{x}^{\boldsymbol{s}_1} \in \mathcal{X}} \\ \max_{\boldsymbol{x}^{\boldsymbol{s}_2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{matrix} \right\} \qquad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Suppose we *don't wait for nature* to select $\boldsymbol{s}$!

Instead, for each possible agent response scenario $\boldsymbol{s}$ …

we find the optimal item to recommend, $\boldsymbol{x}^{\boldsymbol{s}}$…

Aside:

In LPs, discrete vars ($\boldsymbol{s}$) are tricky, so we remove them when possible.

One way to do this is **"indexing"**— rolling out all possible values.

Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \left\{ \begin{array}{c} \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_1} \in \mathcal{X}} \\ \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Fixed

Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \left\{ \begin{array}{c} \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_1} \in \mathcal{X}} \\ \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Fixed

Now, the choice of $\boldsymbol{s}$ doesn't impact the choice of each item!

Elicitation-Recommendation Problem

Reformulation (1): Index by $\boldsymbol{s}$

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \left\{ \begin{array}{c} \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_1} \in \mathcal{X}} \\ \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

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Reformulation (1): Index by $\boldsymbol{s}$

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \left\{ \begin{array}{c} \max\limits_{\boldsymbol{x}^{\boldsymbol{s}^1} \in \mathcal{X}} \\ \max\limits_{\boldsymbol{x}^{\boldsymbol{s}^2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Fixed

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \quad \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \quad \left\{ \begin{array}{c} \max_{\boldsymbol{x}^{\boldsymbol{s}_1}\in\mathcal{X}} \\ \max_{\boldsymbol{x}^{\boldsymbol{s}_2}\in\mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \quad \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Goal:

Express this problem as a finite-size **linear program** (with integer and/or continuous variables)

Reformulation Tricks

1. ~~“Indexing” discrete variables~~
2. Epigraph formulation
3. Linearization
4. Duality
5. Decomposition

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \left\{ \begin{array}{c} \max_{\boldsymbol{x}^{\boldsymbol{s}_1} \in \mathcal{X}} \\ \max_{\boldsymbol{x}^{\boldsymbol{s}_2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \left\{ \begin{array}{c} \max_{\boldsymbol{x}^{\boldsymbol{s}_1}\in\mathcal{X}} \\ \max_{\boldsymbol{x}^{\boldsymbol{s}_2}\in\mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s}\in\mathcal{S}^K \end{array} \right\} \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Q: How can we simplify?

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \left\{ \begin{array}{c} \max\limits_{\boldsymbol{x}^{\boldsymbol{s}^1} \in \mathcal{X}} \\ \max\limits_{\boldsymbol{x}^{\boldsymbol{s}^2} \in \mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Q: How can we simplify?

- Can we select select $\boldsymbol{q}$ and $\boldsymbol{x}^{\boldsymbol{s}}$ simultaneously?

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \left\{ \begin{array}{c} \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_1}\in\mathcal{X}} \\ \max\limits_{\boldsymbol{x}^{\boldsymbol{s}_2}\in\mathcal{X}} \\ \vdots \\ \forall \boldsymbol{s} \in \mathcal{S}^K \end{array} \right\} \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$

Q: How can we simplify?

- Can we select select $\boldsymbol{q}$ and $\boldsymbol{x}^{\boldsymbol{s}}$ simultaneously?

A: Yes. Why?

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$
\begin{array}{lll}
 & \max & \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, & \\
 & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \, : \; \forall \boldsymbol{s} \in \mathcal{S}^K &
\end{array}
$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$
\begin{array}{ll}
\max & \min_{\boldsymbol{s} \in \mathcal{S}^K} \ \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \ \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

Epigraph Formulation of a problem

using aux. variable ($\mathcal{T}$)

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$
\begin{array}{ll}
\max & \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

Epigraph Formulation of a problem

using aux. variable ($\mathcal{T}$)

$$\max_{\boldsymbol{z} \in \mathcal{Z}} \; f(\boldsymbol{z})$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$\begin{array}{ll} \max & \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \end{array}$$

Epigraph Formulation of a problem using aux. variable (τ)

$$\max_{\boldsymbol{z} \in \mathcal{Z}} \; f(\boldsymbol{z}) \quad \Longleftrightarrow \quad \begin{array}{ll} \max\limits_{\tau \in \mathbb{R}, \boldsymbol{z} \in \mathcal{Z}} & \tau \\ & \tau \leq f(\boldsymbol{z}) \end{array}$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$
\begin{array}{ll}
& \max \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

Epigraph Formulation of a problem using aux. variable (τ)

$$
\max_{\boldsymbol{z} \in \mathcal{Z}} \; f(\boldsymbol{z}) \quad \Longleftrightarrow \quad \begin{array}{ll} \max\limits_{\tau \in \mathbb{R}, \, \boldsymbol{z} \in \mathcal{Z}} & \tau \\ & \tau \le f(\boldsymbol{z}) \end{array}
$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$\begin{array}{ll} & \max \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \end{array}$$

Epigraph Formulation of a problem using aux. variable (τ)

$$\max_{\boldsymbol{z} \in \mathcal{Z}} \; f(\boldsymbol{z}) \quad \Longleftrightarrow \quad \begin{array}{ll} \max & \tau \\ \tau \in \mathbb{R}, \boldsymbol{z} \in \mathcal{Z} & \\ \tau \leq f(\boldsymbol{z}) & \end{array}$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$\max \quad \min_{s \in \mathcal{S}^K} \quad \min_{u \in \mathcal{U}(q,s)} \quad u^\top x^s$$

$$\text{s.t.} \quad q \in \mathcal{Q}^K,$$

$$x^s \in \mathcal{X} \; : \; \forall s \in \mathcal{S}^K$$

New Aux. variable! $\boxed{\tau \in \mathbb{R}}$

Epigraph Formulation of a problem using aux. variable ($\mathcal{T}$)

$$\max_{z \in \mathcal{Z}} \; f(z) \quad \Longleftrightarrow \quad \max_{\tau \in \mathbb{R}, z \in \mathcal{Z}} \; \tau \quad \tau \leq f(z)$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$
\begin{aligned}
& \max \quad \boxed{\tau} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \boxed{\tau \in \mathbb{R}}
\end{aligned}
$$

New Aux. variable!

Epigraph Formulation of a problem using aux. variable (τ)

$$
\max_{\boldsymbol{z} \in \mathcal{Z}} \; f(\boldsymbol{z}) \quad \Longleftrightarrow \quad \begin{array}{ll} \max\limits_{\tau \in \mathbb{R}, \boldsymbol{z} \in \mathcal{Z}} & \tau \\ \tau \leq f(\boldsymbol{z}) & \end{array}
$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$
\begin{array}{ll}
 & \max \quad \boxed{\tau} \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
 & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} : \ \forall \boldsymbol{s} \in \mathcal{S}^K \\
 & \boxed{\tau \in \mathbb{R}} \\
 & \boxed{\tau} \leq \min\limits_{\boldsymbol{s} \in \mathcal{S}^K} \ \min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \ \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}
\end{array}
$$

Epigraph Formulation of a problem using aux. variable (τ)

$$
\max_{\boldsymbol{z} \in \mathcal{Z}} \ f(\boldsymbol{z}) \quad \Longleftrightarrow \quad \begin{array}{ll} \max & \tau \\ \tau \in \mathbb{R}, \boldsymbol{z} \in \mathcal{Z} & \\ \tau \leq f(\boldsymbol{z}) & \end{array}
$$

Elicitation-Recommendation Problem

Reformulation (2): Epigraph Form

$$
\begin{array}{ll}
 & \max \quad \tau \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
 & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \, : \; \forall \boldsymbol{s} \in \mathcal{S}^K \\
 & \tau \in \mathbb{R} \\
 & \tau \leq \min\limits_{\boldsymbol{s} \in \mathcal{S}^K} \; \min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}
\end{array}
$$

Epigraph Formulation of a problem using aux. variable (τ)

$$
\max_{\boldsymbol{z} \in \mathcal{Z}} \; f(\boldsymbol{z}) \quad \Longleftrightarrow \quad \begin{array}{ll} \max & \tau \\ \tau \in \mathbb{R}, \boldsymbol{z} \in \mathcal{Z} & \\ \tau \leq f(\boldsymbol{z}) & \end{array}
$$

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} : \ \forall \boldsymbol{s} \in \mathcal{S}^K \\ & \tau \in \mathbb{R} \\ & \tau \leq \displaystyle\min_{\boldsymbol{s} \in \mathcal{S}^K} \ \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \ \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array}$$

Goal:

Express this problem as a finite-size **linear program** (with integer and/or continuous variables)

Reformulation Tricks

1. ~~“Indexing” discrete variables~~
2. ~~Epigraph formulation~~
3. Linearization
4. Duality
5. Decomposition

Elicitation-Recommendation Problem

Reformulation

$$\max_{q \in \mathcal{Q}^K} \quad \min_{s \in \mathcal{S}^K} \quad \max_{x \in \mathcal{X}} \quad \min_{u \in \mathcal{U}(q,s)} \quad u^\top x$$

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & q \in \mathcal{Q}^K, \\ & x^s \in \mathcal{X} \; : \; \forall s \in \mathcal{S}^K \\ & \tau \in \mathbb{R} \\ & \tau \leq \min\limits_{s \in \mathcal{S}^K} \; \min\limits_{u \in \mathcal{U}(q,s)} \; u^\top x^s \end{array}$$

Reformulation Tricks

1. ~~"Indexing" discrete variables~~
2. ~~Epigraph formulation~~
3. Linearization
4. Duality
5. Decomposition

Q: How do we solve this?

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \quad \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \,:\, \forall \boldsymbol{s} \in \mathcal{S}^K \\ & \tau \in \mathbb{R} \\ & \tau \leq \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array}$$

Q: How do we solve this?

- Can we even write this problem down (as a MILP)?

Reformulation Tricks

~~"Indexing" discrete variables~~

~~Epigraph formulation~~

Linearization

4. Duality
5. Decomposition

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \\ & \tau \in \mathbb{R} \\ & \tau \leq \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array}$$

Reformulation Tricks

~~ndexing" discrete variables~~

~~pigraph formulation~~

rization

ty

5. Decomposition

Q: How do we solve this?

- Can we even write this problem down (as a MILP)?

A: No… (s!!)

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \;:\; \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \tau \in \mathbb{R} \\
& \tau \leq \boxed{\min_{\boldsymbol{s} \in \mathcal{S}^K}} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}
\end{aligned}
$$

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
&\max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \tau \in \mathbb{R} \\
& \tau \leq \boxed{\min_{\boldsymbol{s} \in \mathcal{S}^K}} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}
\end{aligned}
$$

Aside:

In LPs, discrete vars ($\boldsymbol{S}$) are tricky, so we remove them when possible.

One way to do this is **"indexing"**— rolling out all possible values.

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} : \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \tau \in \mathbb{R} \\
& \tau \leq \min_{\boldsymbol{s} \in \mathcal{S}^K} \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}
\end{aligned}
$$

Aside:

In LPs, discrete vars ($\boldsymbol{S}$) are tricky, so we remove them when possible.

One way to do this is **"indexing"**— rolling out all possible values.

Linearizing a Min constraint

by "rolling out" all possible outcomes

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \tau \in \mathbb{R} \\
& \tau \leq \boxed{\min_{\boldsymbol{s} \in \mathcal{S}^K}} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}
\end{aligned}
$$

Aside:

In LPs, discrete vars ($\boldsymbol{S}$) are tricky, so we remove them when possible.

One way to do this is **"indexing"**— rolling out all possible values.

Linearizing a Min constraint

by "rolling out" all possible outcomes

$$
\begin{aligned}
& \max \quad \tau \\
& \tau \in \mathbb{R} \\
& \tau \leq \min_{\boldsymbol{y} \in \mathcal{Y}} f(\boldsymbol{y})
\end{aligned}
$$

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$\begin{aligned} \max \quad & \tau \\ \text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} : \ \forall \boldsymbol{s} \in \mathcal{S}^K \\ & \tau \in \mathbb{R} \\ & \tau \leq \min_{\boldsymbol{s} \in \mathcal{S}^K} \ \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \ \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{aligned}$$

Aside:

In LPs, discrete vars ($\boldsymbol{S}$) are tricky, so we remove them when possible.

One way to do this is **"indexing"**— rolling out all possible values.

Linearizing a Min constraint

by "rolling out" all possible outcomes

$$\begin{aligned} \max \quad & \tau \\ & \tau \in \mathbb{R} \\ & \tau \leq \min_{\boldsymbol{y} \in \mathcal{Y}} f(\boldsymbol{y}) \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max \quad & \tau \\ & \tau \in \mathbb{R} \\ & \tau \leq f(\boldsymbol{y}) \quad \forall \boldsymbol{y} \in \mathcal{Y} \end{aligned}$$

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$\begin{aligned} \max \quad & \tau \\ \text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \, : \, \forall \boldsymbol{s} \in \mathcal{S}^K \\ & \tau \in \mathbb{R} \\ & \tau \leq \boxed{\min_{\boldsymbol{s} \in \mathcal{S}^K}} \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{aligned}$$

Aside:

In LPs, discrete vars ($\boldsymbol{S}$) are tricky, so we remove them when possible.

One way to do this is **"indexing"**— rolling out all possible values.

Linearizing a Min constraint

by "rolling out" all possible outcomes

$$\begin{aligned} \max \quad & \tau \\ & \tau \in \mathbb{R} \\ & \tau \leq \min_{\boldsymbol{y} \in \mathcal{Y}} f(\boldsymbol{y}) \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max \quad & \tau \\ & \tau \in \mathbb{R} \\ & \tau \leq f(\boldsymbol{y}) \;\; \forall \boldsymbol{y} \in \mathcal{Y} \end{aligned}$$

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \tau \in \mathbb{R} \\
& \tau \leq \min_{\boldsymbol{s} \in \mathcal{S}^K} \; \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}
\end{aligned}
$$

Linearizing a Min constraint

$$
\begin{aligned}
\max \quad & \tau \\
& \tau \in \mathbb{R} \\
& \tau \leq \min_{\boldsymbol{y} \in \mathcal{Y}} f(\boldsymbol{y})
\end{aligned}
\quad \Longleftrightarrow \quad
\begin{aligned}
\max \quad & \tau \\
& \tau \in \mathbb{R} \\
& \tau \leq f(\boldsymbol{y}) \;\; \forall \boldsymbol{y} \in \mathcal{Y}
\end{aligned}
$$

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \, : \, \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \tau \in \mathbb{R} \\
& \tau \leq \min_{\boldsymbol{s} \in \mathcal{S}^K} \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Linearizing a Min constraint

$$
\begin{aligned}
\max \quad & \tau \\
& \tau \in \mathbb{R} \\
& \tau \leq \min_{\boldsymbol{y} \in \mathcal{Y}} f(\boldsymbol{y})
\end{aligned}
\quad \Longleftrightarrow \quad
\begin{aligned}
\max \quad & \tau \\
& \tau \in \mathbb{R} \\
& \tau \leq f(\boldsymbol{y}) \quad \forall \boldsymbol{y} \in \mathcal{Y}
\end{aligned}
$$

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \tau \in \mathbb{R} \\
& \tau \leq \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
& \max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \; : \; \forall \boldsymbol{s} \in \mathcal{S}^K \\
& \tau \in \mathbb{R} \\
& \tau \leq \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Comment:

Constraints/vars group naturally by $\boldsymbol{s}$

Elicitation-Recommendation Problem

Reformulation (3): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
& \max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \le \min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Comment:

Constraints/vars group naturally by $\boldsymbol{s}$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
& \max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{aligned} & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ & \tau \leq \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{aligned} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \boxed{\min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})}} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

The only remaining problem is the minimization over $\boldsymbol{u}$.

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left.\begin{cases} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \displaystyle\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \; \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{cases}\right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

The only remaining problem is the minimization over $\boldsymbol{u}$.

Q: How do we get rid of this minimization?

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{array}{ll}
\max & \tau \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left.\begin{cases} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{cases}\right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

The only remaining problem is the minimization over $\boldsymbol{u}$.

Q: How do we get rid of this minimization?

… what would happen if this *min* became a *max*?

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \tau \in \mathbb{R} \\ & \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K \end{array}$$

The only remaining problem is the minimization over $\boldsymbol{u}$.

Primal & Dual Linear Programs

$$\min_{\boldsymbol{x} \in \mathbb{R}^N, \; A\boldsymbol{x} \geq \boldsymbol{b}} \boldsymbol{c}^\top \boldsymbol{x}$$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{array}{ll}
 & \max \quad \tau \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
 & \tau \in \mathbb{R} \\
 & \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \displaystyle\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

The only remaining problem is the minimization over $\boldsymbol{u}$.

Primal & Dual Linear Programs

$$
\min_{\boldsymbol{x} \in \mathbb{R}^N, \ A\boldsymbol{x} \geq \boldsymbol{b}} \boldsymbol{c}^\top \boldsymbol{x} \quad \Longleftrightarrow \quad \max_{\boldsymbol{y} \in \mathbb{R}^M, \ A^\top \boldsymbol{y} \leq \boldsymbol{c}} \boldsymbol{b}^\top \boldsymbol{y}
$$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
&\max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \displaystyle\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Primal & Dual Linear Programs

$$
\min_{\boldsymbol{x} \in \mathbb{R}^N, \;\; A\boldsymbol{x} \geq \boldsymbol{b}} \boldsymbol{c}^\top \boldsymbol{x} \quad \Longleftrightarrow \quad \max_{\boldsymbol{y} \in \mathbb{R}^M, \;\; A^\top \boldsymbol{y} \leq \boldsymbol{c}} \boldsymbol{b}^\top \boldsymbol{y}
$$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$\min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$ "primal"

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \tau \in \mathbb{R} \\ & \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \min\limits_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K \end{array}$$

Primal & Dual Linear Programs

$$\min_{\boldsymbol{x}\in\mathbb{R}^N,\ A\boldsymbol{x}\geq\boldsymbol{b}} \boldsymbol{c}^\top \boldsymbol{x} \quad \Longleftrightarrow \quad \max_{\boldsymbol{y}\in\mathbb{R}^M,\ A^\top\boldsymbol{y}\leq\boldsymbol{c}} \boldsymbol{b}^\top \boldsymbol{y}$$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \quad \text{“primal”}$$

$$\begin{array}{ll} \max & \tau \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \tau \in \mathbb{R} \\ & \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \displaystyle\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K \end{array}$$

Primal & Dual Linear Programs

$$\mathcal{U}(\boldsymbol{q}, \boldsymbol{s}) := \left\{ \begin{array}{lll} \boldsymbol{u} \in \mathcal{U}^0 & : & \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}^A_\kappa} - \boldsymbol{x}^{\boldsymbol{q}^B_\kappa}) \geq 0 \quad \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = 1 \\ & & \boldsymbol{u}^\top (\boldsymbol{x}^{\boldsymbol{q}^A_\kappa} - \boldsymbol{x}^{\boldsymbol{q}^B_\kappa}) \leq 0 \quad \forall \kappa = 1, \ldots, K : \ \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$\begin{array}{ll} & \max \quad \tau \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \tau \in \mathbb{R} \\ & \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K \end{array}$$

$$\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$ “primal”

$\Updownarrow$

$$\max_{\boldsymbol{y} \in \mathcal{Y}} \boldsymbol{y}^\top \boldsymbol{b}$$ “dual”

Primal & Dual Linear Programs

$$\mathcal{U}(\boldsymbol{q},\boldsymbol{s}) := \left\{ \boldsymbol{u} \in \mathcal{U}^0 \;:\; \begin{array}{ll} \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \geq 0 & \forall \kappa = 1,\ldots,K : \;\; \boldsymbol{s}_\kappa = 1 \\ \boldsymbol{u}^\top(\boldsymbol{x}^{\boldsymbol{q}_\kappa^A} - \boldsymbol{x}^{\boldsymbol{q}_\kappa^B}) \leq 0 & \forall \kappa = 1,\ldots,K : \;\; \boldsymbol{s}_\kappa = -1 \end{array} \right\}$$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$\begin{aligned} \max \quad & \tau \\ \text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \tau \in \mathbb{R} \\ & \left\{ \begin{aligned} & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ & \tau \leq \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{aligned} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K \end{aligned}$$

$$\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$ “primal”

$$\Updownarrow$$

$$\max_{\boldsymbol{y} \in \mathcal{Y}} \; \boldsymbol{y}^\top \boldsymbol{b}$$ “dual”

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{array}{ll}
\max & \tau \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \displaystyle\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

$$\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$ “primal”

$\Updownarrow$

$$\max_{\boldsymbol{y} \in \mathcal{Y}} \boldsymbol{y}^\top \boldsymbol{b}$$ “dual”

Q: How many different primal/dual problems are there?

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$\begin{aligned} \max \quad & \tau \\ \text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \tau \in \mathbb{R} \\ & \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \min\limits_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K \end{aligned}$$

$$\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$ "primal"

$\Updownarrow$

$$\max_{\boldsymbol{y} \in \mathcal{Y}} \boldsymbol{y}^\top \boldsymbol{b}$$ "dual"

Q: How many different primal/dual problems are there?

A: One for each $\boldsymbol{s} \in \mathcal{S}^K$…

Because we "indexed" over these variables, we need to create different variables for each.

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{array}{ll}
\max & \tau \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \displaystyle\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}} \end{array} \right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

$$\min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \boldsymbol{u}^\top \boldsymbol{x}^{\boldsymbol{s}}$$ “primal”

$$\max_{\boldsymbol{y} \in \mathcal{Y}} \boldsymbol{y}^\top \boldsymbol{b}$$ “dual”

$$\max_{\boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s},\boldsymbol{x}^{\boldsymbol{s}},\boldsymbol{q})} (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}}$$

Q: How many different primal/dual problems are there?

A: One for each $\boldsymbol{s} \in \mathcal{S}^K$**…**

Because we “indexed” over these variables, we need to create different variables for each.

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
& \max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{
\begin{array}{l}
\boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\
\tau \leq \max\limits_{\boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q})} (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}}
\end{array}
\right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \displaystyle\max_{\boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q})} (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Q: How do we get rid of this?

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{array}{ll}
\max & \tau \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \tau \leq \max\limits_{\boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q})} (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

Q: How do we get rid of this?

A: Add $\boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q})$ to the domain…

Elicitation-Recommendation Problem

Reformulation (4): Index by $\boldsymbol{s}$… again!

$$
\begin{aligned}
& \max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{
\begin{array}{l}
\boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\
\boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\
\tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}}
\end{array}
\right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q}, \boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\begin{aligned} \max \quad & \tau \\ \text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \tau \in \mathbb{R} \\ & \left\{ \begin{aligned} & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ & \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ & \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{aligned} \right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K \end{aligned}$$

Goal:

Express this problem as a finite-size **linear program** (with integer and/or continuous variables)

Reformulation Tricks

1. ~~"Indexing" discrete variables~~
2. ~~Epigraph formulation~~
3. ~~Linearization~~
4. ~~Duality~~
5. Decomposition

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \quad \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\begin{array}{lll} & \max & \tau \\ \text{s.t.} & & \boldsymbol{q} \in \mathcal{Q}^K, \\ & & \tau \in \mathbb{R} \\ & & \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K \end{array}$$

- Can we solve this yet???

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q} \in \mathcal{Q}^K} \quad \min_{\boldsymbol{s} \in \mathcal{S}^K} \quad \max_{\boldsymbol{x} \in \mathcal{X}} \quad \min_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\begin{array}{lll} \max & \tau & \\ \text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, & \\ & \tau \in \mathbb{R} & \\ & \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} & \forall \boldsymbol{s} \in \mathcal{S}^K \end{array}$$

- Can we solve this yet???
- The number of response scenarios $\boldsymbol{s} \in \mathcal{S}^K$ can be huge

Elicitation-Recommendation Problem

Reformulation

$$\max_{\boldsymbol{q}\in\mathcal{Q}^K} \quad \min_{\boldsymbol{s}\in\mathcal{S}^K} \quad \max_{\boldsymbol{x}\in\mathcal{X}} \quad \min_{\boldsymbol{u}\in\mathcal{U}(\boldsymbol{q},\boldsymbol{s})} \quad \boldsymbol{u}^\top \boldsymbol{x}$$

$$\begin{aligned} \max \quad & \tau \\ \text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\ & \tau \in \mathbb{R} \\ & \left\{ \begin{aligned} & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ & \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ & \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{aligned} \right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K \end{aligned}$$

- Can we solve this yet???
- The number of response scenarios $\boldsymbol{s} \in \mathcal{S}^K$ can be huge
- We address this with **decomposition**

Elicitation-Recommendation Problem

Decomposition

$$
\begin{array}{ll}
 & \max \quad \tau \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, \\
 & \tau \in \mathbb{R} \\
 & \left.\begin{cases} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{cases}\right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{array}
$$

Elicitation-Recommendation Problem

Decomposition

$$
\begin{aligned}
& \max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left.\begin{cases} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{cases}\right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

General Idea:

Elicitation-Recommendation Problem

Decomposition

$$
\begin{aligned}
&\max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left.\begin{cases} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{cases}\right\} \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

General Idea:

1. start with only a few scenarios:
$$\mathcal{S}' := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$$

Elicitation-Recommendation Problem

Decomposition

$$
\begin{aligned}
& \max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} \quad \forall \boldsymbol{s} \in \mathcal{S}^K
\end{aligned}
$$

General Idea:

1. start with only a few scenarios:
$$\mathcal{S}' := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$$
2. solve this **reduced problem**

Elicitation-Recommendation Problem

Decomposition

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{aligned} & \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ & \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ & \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{aligned} \right\} \quad \cancel{\forall \boldsymbol{s} \in \mathcal{S}^K} \;\; \forall \boldsymbol{s} \in \mathcal{S}'
\end{aligned}
$$

General Idea:

1. start with only a few scenarios:
$$\mathcal{S}' := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$$

2. solve this **reduced problem**

Elicitation-Recommendation Problem

Decomposition

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \le (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} \quad \cancel{\forall \boldsymbol{s} \in \mathcal{S}^K} \;\; \forall \boldsymbol{s} \in \mathcal{S}'
\end{aligned}
$$

General Idea:

1. start with only a few scenarios:
   $$\mathcal{S}' := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$$
2. solve this **reduced problem**
3. Are there any violated **constraints** for $\forall \boldsymbol{s} \in \mathcal{S}^K$ (is $\mathcal{T}$ too large?)

Elicitation-Recommendation Problem

Decomposition

$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{
\begin{array}{l}
\boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\
\boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\
\tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}}
\end{array}
\right\} \quad \cancel{\forall \boldsymbol{s} \in \mathcal{S}^K} \;\; \forall \boldsymbol{s} \in \mathcal{S}'
\end{aligned}
$$

Q: When is a feasible solution to the **reduced problem** also a feasible solution to the **"master"** problem?

General Idea:

1. start with only a few scenarios:
   $$\mathcal{S}' := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$$
2. solve this **reduced problem**
3. Are there any violated **constraints** for $\forall \boldsymbol{s} \in \mathcal{S}^K$ (is $\mathcal{T}$ too large?)

Elicitation-Recommendation Problem

Decomposition

$$
\begin{aligned}
& \max \quad \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} \quad \cancel{\forall \boldsymbol{s} \in \mathcal{S}^K} \quad \forall \boldsymbol{s} \in \mathcal{S}'
\end{aligned}
$$

General Idea:

1. start with only a few scenarios:
   $$\mathcal{S}' := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$$
2. solve this **reduced problem**
3. Are there any violated **constraints** for $\forall \boldsymbol{s} \in \mathcal{S}^K$ (is τ too large?)
4. **If No:** we're optimal!

Q: When is a feasible solution to the **reduced problem** also a feasible solution to the **"master"** problem?

Elicitation-Recommendation Problem

Decomposition

$$
\begin{array}{rll}
\max & \tau & \\
\text{s.t.} & \boldsymbol{q} \in \mathcal{Q}^K, & \\
& \tau \in \mathbb{R} & \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} & \cancel{\forall \boldsymbol{s} \in \mathcal{S}^K} \\
& & \forall \boldsymbol{s} \in \mathcal{S}'
\end{array}
$$

General Idea:

1. start with only a few scenarios:
   $$\mathcal{S}' := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$$
2. solve this **reduced problem**
3. Are there any violated **constraints** for $\forall \boldsymbol{s} \in \mathcal{S}^K$ (is τ too large?)
4. **If No:** we're optimal!

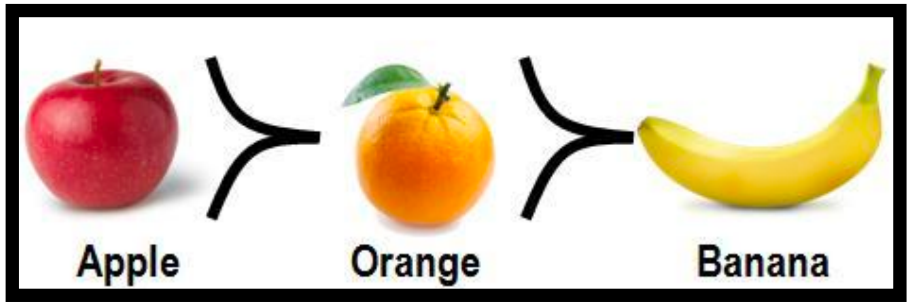
Elicitation-Recommendation Problem

Decomposition

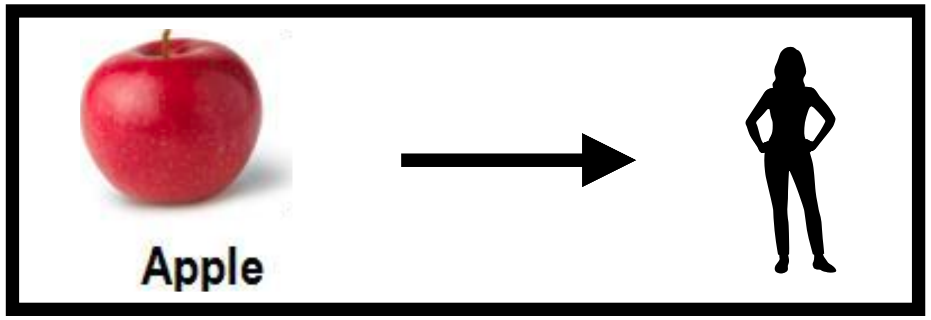
$$
\begin{aligned}
\max \quad & \tau \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{Q}^K, \\
& \tau \in \mathbb{R} \\
& \left\{ \begin{array}{l} \boldsymbol{x}^{\boldsymbol{s}} \in \mathcal{X} \\ \boldsymbol{y}^{\boldsymbol{s}} \in \mathcal{Y}(\boldsymbol{s}, \boldsymbol{x}^{\boldsymbol{s}}, \boldsymbol{q}) \\ \tau \leq (\boldsymbol{b}^{\boldsymbol{s}})^\top \boldsymbol{y}^{\boldsymbol{s}} \end{array} \right\} \quad \cancel{\forall \boldsymbol{s} \in \mathcal{S}^K} \\
& \qquad \forall \boldsymbol{s} \in \mathcal{S}'
\end{aligned}
$$

General Idea:

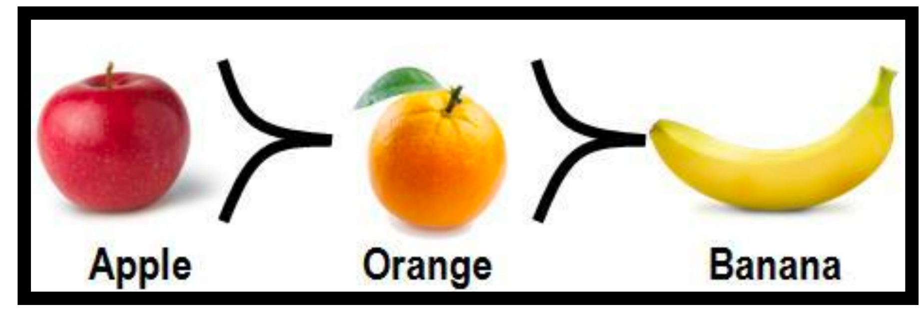
1. start with only a few scenarios:
   $$\mathcal{S}' := \{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$$
2. solve this **reduced problem**
3. Are there any violated **constraints** for $\forall \boldsymbol{s} \in \mathcal{S}^K$ (is τ too large?)
4. **If No:** we're optimal!
5. **If Yes:** Add some violated scenarios to $\mathcal{S}'$, repeat

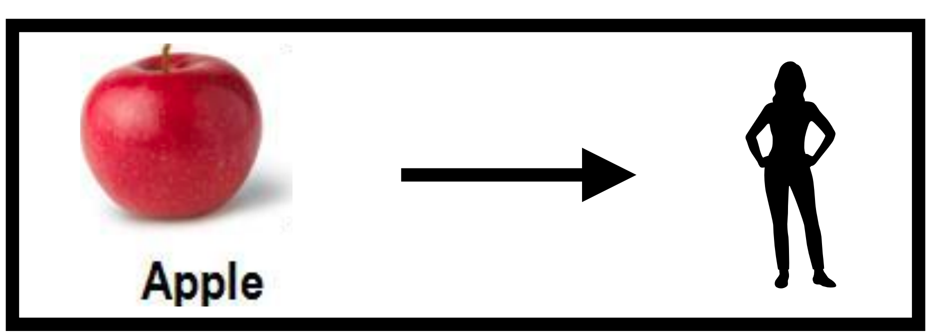


What’s Next?



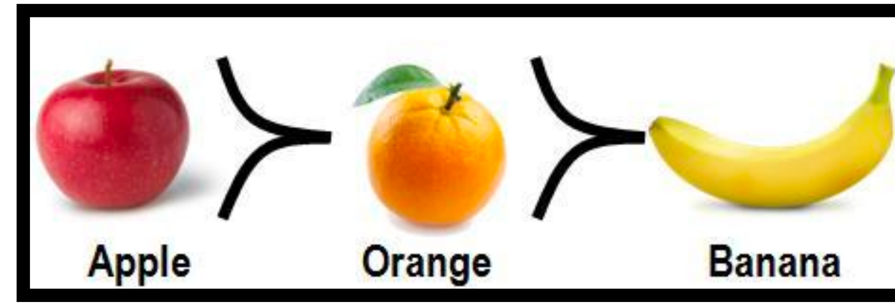
Potential Research Questions



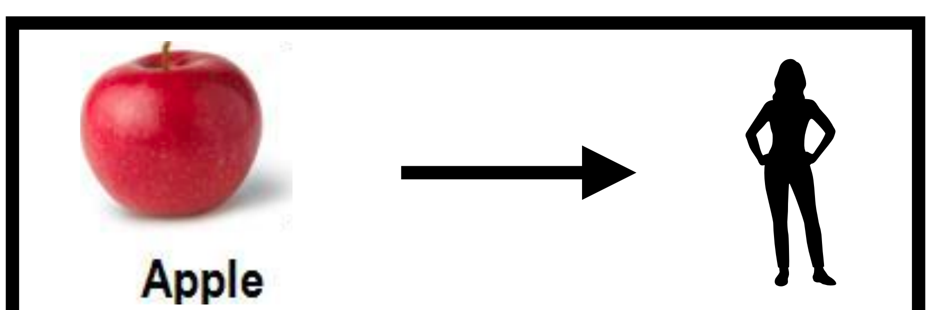


Potential Research Questions

Theory

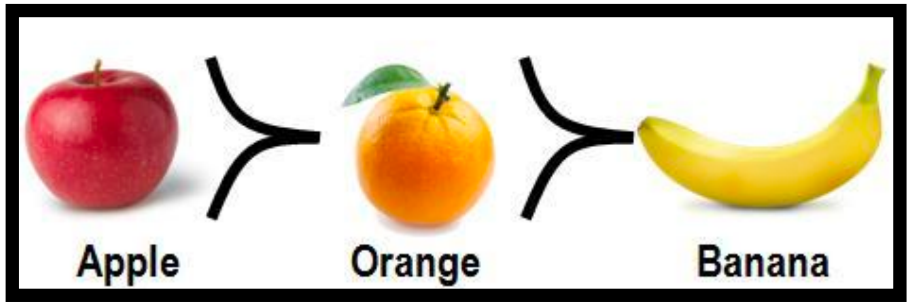


Application



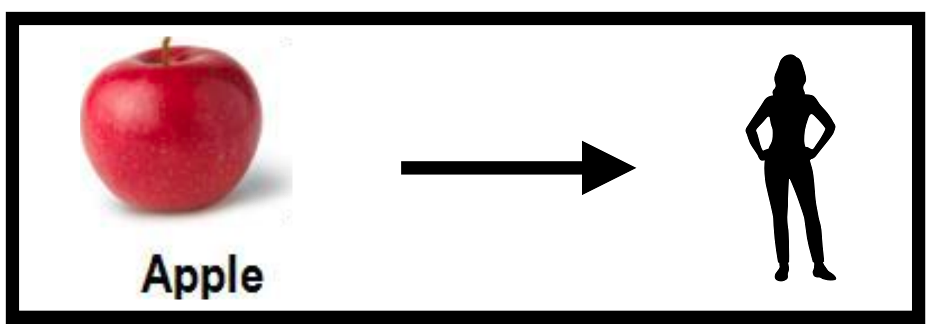
Potential Research Questions

Theory



Application

Q1. Can we learn preferences in a **lab setting**?
(prefs for classes? brands? music?)



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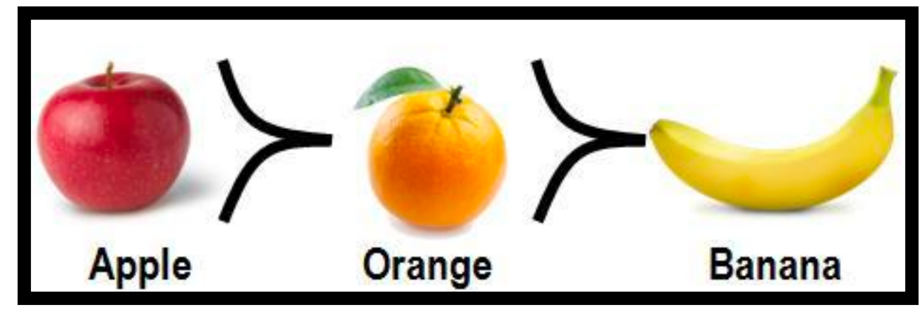

Application

Q1. Can we learn preferences in a **lab setting**?
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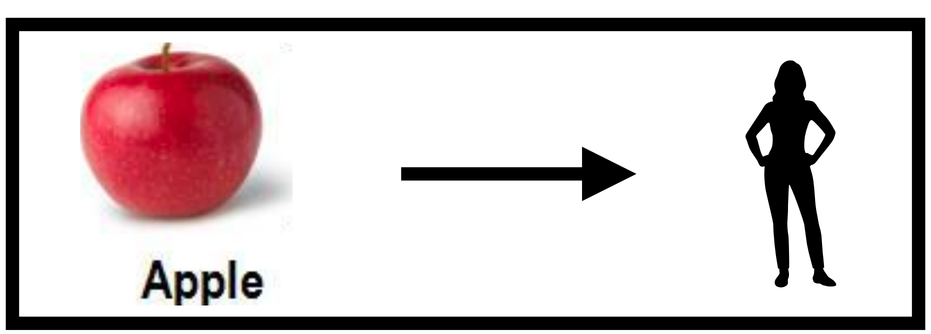
Q2. Do our **preference assumptions** hold?
- transitivity
- consistency



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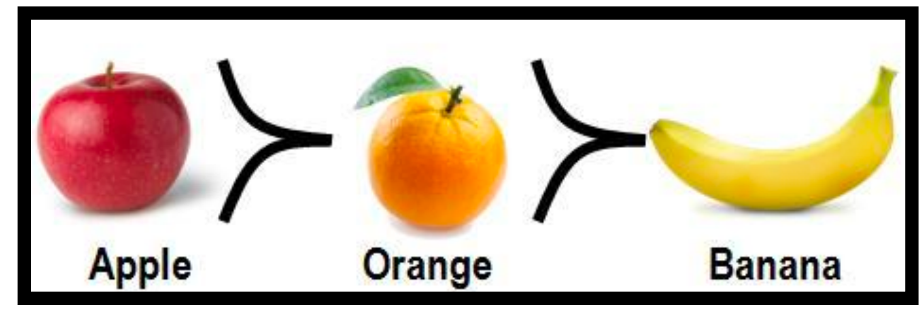

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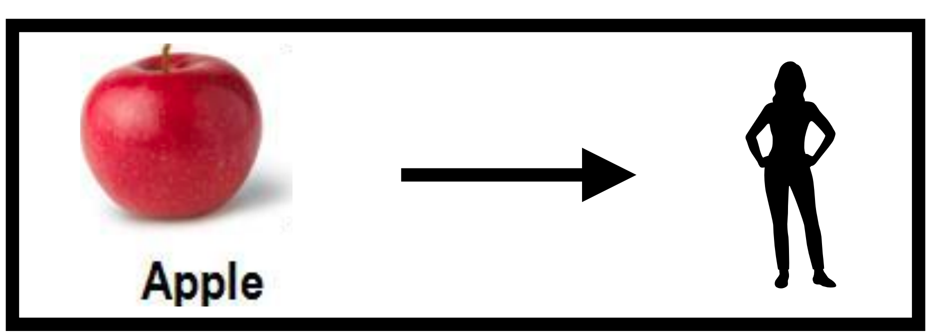
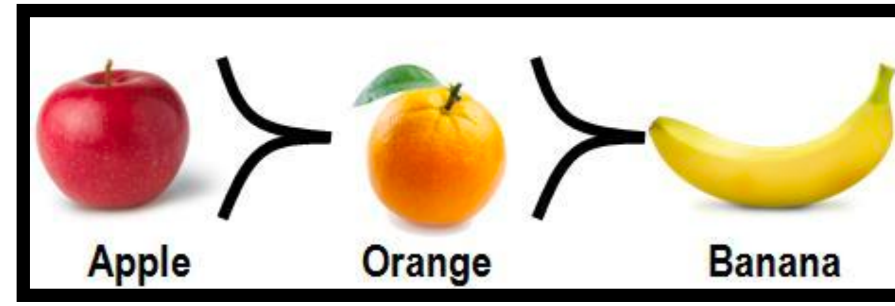
Q3. Join elicitation with a **decision process**?
(hiring, resource allocation)



Potential Research Questions

Theory

Q4. **How many queries** are needed for an "optimal" recommendation?
(can we ever guarantee rec. quality?)



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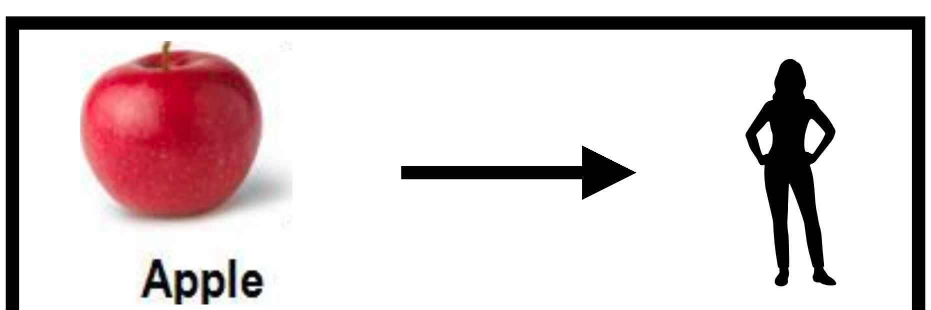

Potential Research Questions

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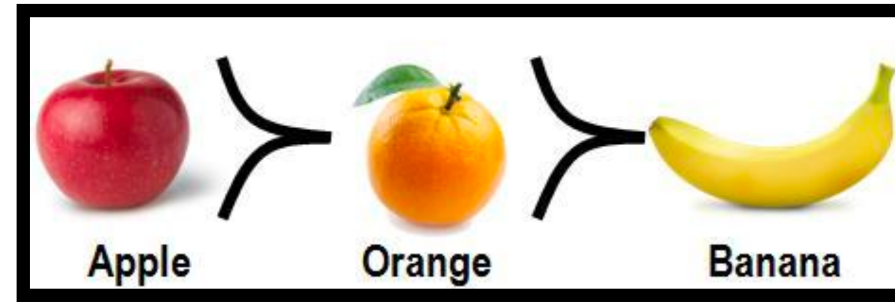

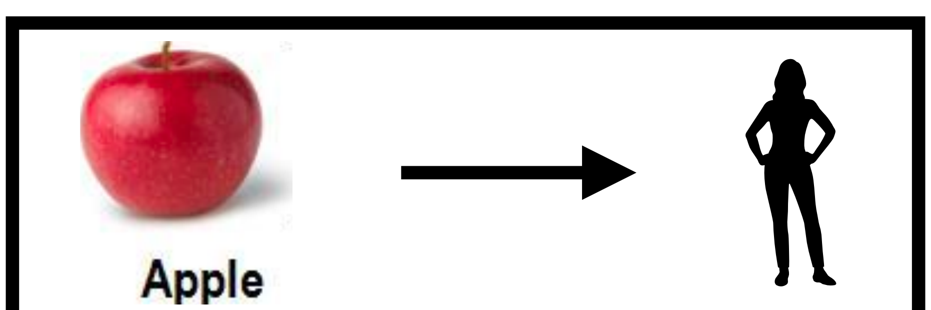

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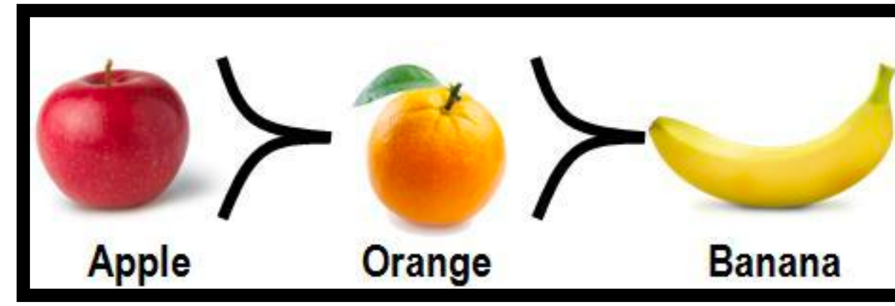
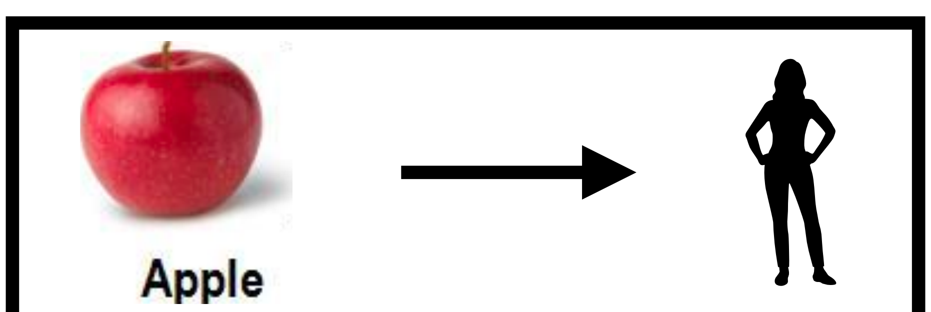
Q6. What if we have a **group** of agents?
(cc: social choice?)





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